lektion12

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1 Lektion 12

[1]:
```
import matplotlib.pyplot as plt
import numpy as np
from sympy import *

init_printing()
%matplotlib inline
```

1.1 Gradient, Hessematrix (Wiederholung)

[2]:
```
x, y, z = symbols('x y z')
var = (x, y, z)
```

[3]:
```
f = Function('f')
f(*var)
```

[3]:
$f(x, y, z)$

[4]:
```
G = Matrix([f(*var)]).jacobian(var)
G
```

[4]:
$$\left[\frac{\partial}{\partial x}f(x,y,z) \quad \frac{\partial}{\partial y}f(x,y,z) \quad \frac{\partial}{\partial z}f(x,y,z)\right]$$

[5]:
```
H = hessian(f(*var), var)
H
```

[5]:
$$\begin{bmatrix} \frac{\partial^2}{\partial x^2}f(x,y,z) & \frac{\partial^2}{\partial y \partial x}f(x,y,z) & \frac{\partial^2}{\partial z \partial x}f(x,y,z) \\ \frac{\partial^2}{\partial y \partial x}f(x,y,z) & \frac{\partial^2}{\partial y^2}f(x,y,z) & \frac{\partial^2}{\partial z \partial y}f(x,y,z) \\ \frac{\partial^2}{\partial z \partial x}f(x,y,z) & \frac{\partial^2}{\partial z \partial y}f(x,y,z) & \frac{\partial^2}{\partial z^2}f(x,y,z) \end{bmatrix}$$

[6]:
```
Matrix([f(*var)]).hessian(var)
```

```
---------------------------------------------------------------------------
AttributeError                            Traceback (most recent call last)
Cell In[6], line 1
----> 1 Matrix([f(*var)]).hessian(var)

AttributeError: 'MutableDenseMatrix' object has no attribute 'hessian'
```

[7]:
```
H = Matrix([f(*var)]).jacobian(var).jacobian(var)
H
```

[7]:
$$\begin{bmatrix} \frac{\partial^2}{\partial x^2}f(x,y,z) & \frac{\partial^2}{\partial y \partial x}f(x,y,z) & \frac{\partial^2}{\partial z \partial x}f(x,y,z) \\ \frac{\partial^2}{\partial y \partial x}f(x,y,z) & \frac{\partial^2}{\partial y^2}f(x,y,z) & \frac{\partial^2}{\partial z \partial y}f(x,y,z) \\ \frac{\partial^2}{\partial z \partial x}f(x,y,z) & \frac{\partial^2}{\partial z \partial y}f(x,y,z) & \frac{\partial^2}{\partial z^2}f(x,y,z) \end{bmatrix}$$

1.2 Beispiel

[8]:
```
var = (x, y)
f = 2 - x**2 / 2 - y**2
f
```

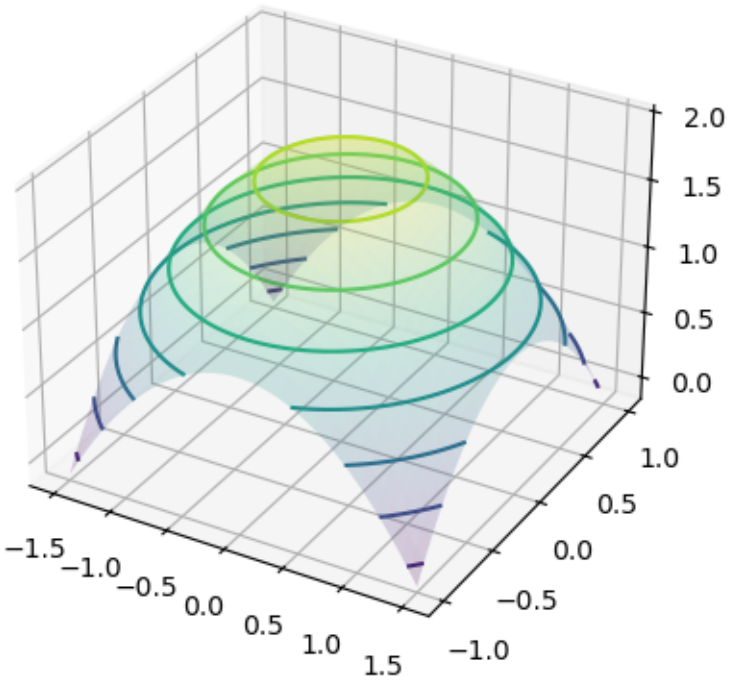
[8]:
$$-\frac{x^2}{2} - y^2 + 2$$

[9]:
```
fn = lambdify(var, f)
```

[10]:
```
xn = np.linspace(-1.5, 1.5, 91)
yn = np.linspace(-1, 1, 61)
X, Y = np.meshgrid(xn, yn)
F = fn(X, Y)
```

[11]:
```
fig1 = plt.figure()
ax1 = fig1.add_subplot(projection='3d')
ax1.plot_surface(X, Y, F, alpha=.2, cmap=plt.cm.viridis)
ax1.contour(X, Y, F)
```

[11]: <mpl_toolkits.mplot3d.art3d.QuadContourSet3D at 0x770c555d7b60>



1.2.1 Extrema (Wiederholung)

[12]:
```
ext = solve(Matrix([f]).jacobian(var))
ext
```

[12]: $\{x : 0,\ y : 0\}$

[13]:
```
H = hessian(f, var)
H
```

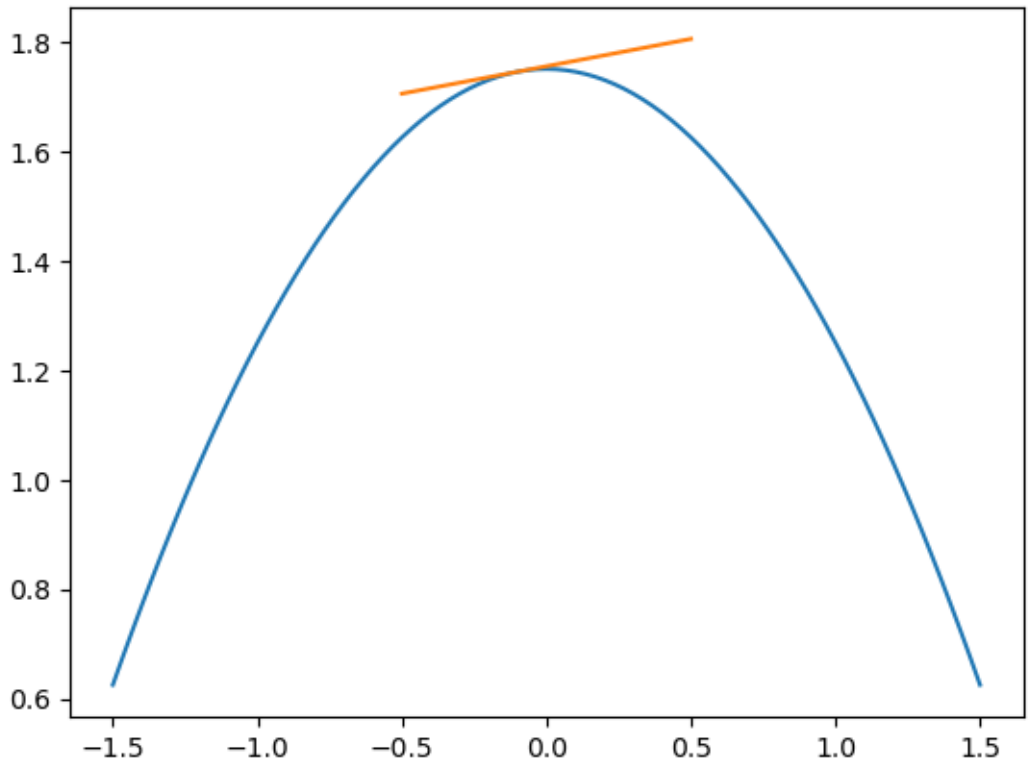
[13]: $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

1.2.2 Tangenten

[14]:
```
fig1d = plt.figure()
ax1d = fig1d.gca()

pnt = {x: -.1, y: -.5}


def tangente1dx(f, pnt):
```

```
    """ Parametrisierung der Tangeten an f in pnt in x-Richtung
    """
    t = np.linspace(-.5, .5)
    return t, f.subs(pnt) + f.diff(x).subs(pnt) * (t - pnt[x])


ax1d.plot(xn, fn(xn, pnt[y]))
ax1d.plot(*tangente1dx(f, pnt))
```

[14]: [<matplotlib.lines.Line2D at 0x770c53e9b290>]

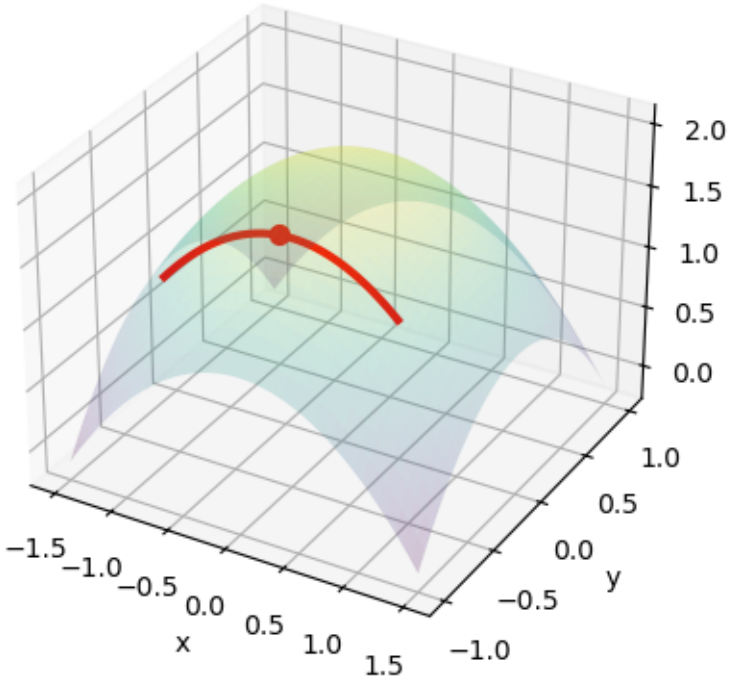


[15]:
```
t = symbols('t')
fdxn = lambdify((x, y), f.diff(x))
xn_c = xn[2:-2:8]

ax1d.quiver(xn_c,
            fn(xn_c, pnt[y]),
            1 + 0 * xn_c,
            fdxn(xn_c, pnt[y]),
            angles='xy', scale_units='xy', scale=3, color='purple')
ax1d.scatter(xn_c, fn(xn_c, pnt[y]), c='red')
```

[15]: <matplotlib.collections.PathCollection at 0x770c53be9f10>

[16]:
```
fig1 = plt.figure()
ax1 = fig1.add_subplot(111, projection='3d')
ax1.plot_surface(X, Y, F, cmap=plt.cm.viridis, alpha=0.2)
ax1.set_xlabel('x')
ax1.set_ylabel('y')

vec = [1, 0]


def cut(pnt, f, vec, scale=1):
    '''
    Schnittkurve durch Punkt pnt in Richtung vec
    '''
    tn = np.linspace(-1, 1) / scale
    xn = pnt[x] + vec[0] * tn
    yn = pnt[y] + vec[1] * tn
    zn = lambdify((x, y), f)(xn, yn)
    return xn, yn, zn


ax1.scatter(pnt[x], pnt[y], f.subs(pnt), c='r', s=50)
ax1.plot(*cut(pnt, f, vec), 'r', lw=3)
```

[16]: [<mpl_toolkits.mplot3d.art3d.Line3D at 0x770c53bc6a50>]



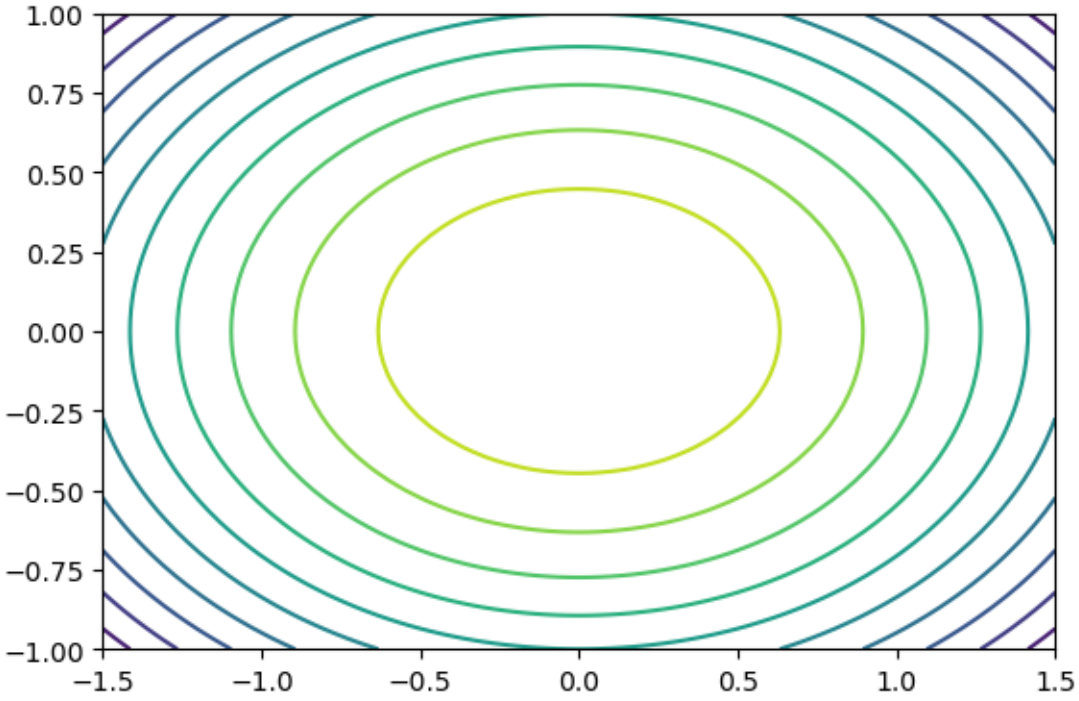
[17]:
```
def tangente2(pnt, f, vec, var, scale=1):
    '''
    Tangente an Punkt (pnt, f(pnt)) in Richtung vec
    als parametrische Kurve
    '''
    t_ = symbols('t')
    fc = (f + t_ *
          (Matrix([f]).jacobian(var).dot(Matrix(2, 1, vec)))).subs(pnt)
    fcn = lambdify(t_, fc)

    tn = np.linspace(-1, 1) / scale
    xn = pnt[x] + vec[0] * tn
    yn = pnt[y] + vec[1] * tn
    zn = fcn(tn)

    return xn, yn, zn


ax1.plot(*tangente2(pnt, f, vec, var), 'c', lw=3)
```

[17]:
```
[<mpl_toolkits.mplot3d.art3d.Line3D at 0x770c53cbb620>]
```

1.2.3 Gradient als Vektorfeld (quiver) Höhenlinien (contour)

[18]:
```
X, Y = np.meshgrid(xn, yn)
X_c = X[5:-2:7, 5:-2:7]
Y_c = Y[5:-2:7, 5:-2:7]
grad = Matrix([f]).jacobian((x, y))
gradn = lambdify((x, y), grad)
GR = gradn(X_c, Y_c)[0]
```

[19]:
```
fig = plt.figure()
ax = fig.gca()
ax.contour(X, Y, fn(X, Y), 10)
ax.set_aspect('equal', 'box')
```



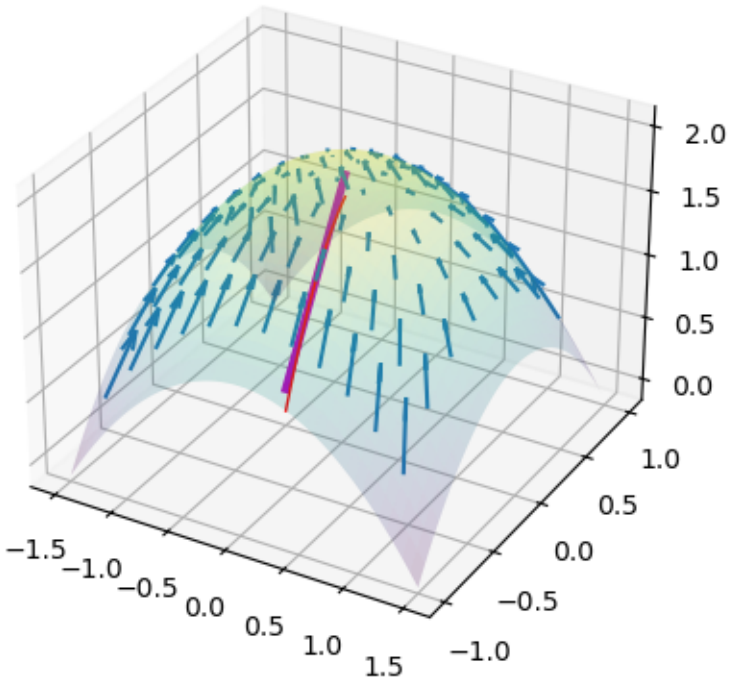
[20]:
```
ax.quiver(X_c, Y_c, GR[0], GR[1], angles='xy', scale_units='xy', scale=8,␣
↪color='red');
```

[21]:
```
t_, dx, dy = symbols('t dx dy')
fc = f.subs({x: x + t_ * dx, y: y + t_ * dy})
dfc = diff(fc, t_)
dfcn = lambdify((x, y, dx, dy, t_), dfc)
```

[22]:
```
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
```

```
ax.plot_surface(X, Y, fn(X, Y), cmap=plt.cm.viridis, alpha=.2)

zG = dfcn(X_c, Y_c, GR[0, :, :], GR[1, :, :], 0)

ax.quiver(X_c, Y_c, fn(X_c, Y_c), GR[0, :, :] / 10,
          GR[1, :, :] / 10, zG / 10)


pnt = {x: X_c[1, 7], y: Y_c[1, 7]}
ax.plot(*tangente2(pnt, f, GR[:, 1, 7], var=var, scale=3), 'm', lw=3)
ax.plot(*cut(pnt, f, GR[:, 1, 7], scale=3), 'r', lw=1)
```

[22]: `[<mpl_toolkits.mplot3d.art3d.Line3D at 0x770c521bd0d0>]`



1.3 Differentialgleichungen

vgl. Analysis II

1.3.1 erstes Beispiel

[23]:
```
import matplotlib.pyplot as plt
import numpy as np
from sympy import *
init_printing()
%matplotlib notebook
```

Gesucht ist eine differenzierbare Funktion $y : \mathbb{R} \to \mathbb{R}$ mit

$$\frac{d}{dt}y(t) = y(t).$$

[24]:
```
y = Function('y')
t, tau = symbols('t tau', real = True)
dgl = Eq(y(t).diff(t), y(t))
dgl
```

[24]:
$$\frac{d}{dt}y(t) = y(t)$$

[25]:
```
sol = dsolve(dgl, y(t))
sol
```

[25]:
$$y(t) = C_1 e^t$$

[26]:
```
sol.subs(symbols('C1'), 1)
```

[26]:
$$y(t) = e^t$$

[27]:
```
c1_s=symbols('C1')
```

Wir ergänzen die Differentialgleichunng mit einer Anfangsbedingung zur einem Anfangswertproblem.

Gesucht ist $y : \mathbb{R} \to \mathbb{R}$ mit

$$\begin{aligned} \frac{d}{dt}y(t) &= y(t) \\ y(t_0) &= y_0 \end{aligned}$$

[28]:
```
t0, y0 = symbols('t_0 y_0')
eq = sol.subs({t: t0}).subs({y(t0): y0})
eq
```

[28]:
$$y_0 = C_1 e^{t_0}$$

[29]:
```
sol = sol.subs(c1_s, solve(eq, c1_s)[0])
sol
```

[29]:
$$y(t) = y_0 e^t e^{-t_0}$$

[30]:
```
checkodesol(dgl, sol)
```

[30]:
```
(True, 0)
```

Das geht auch einfacher

[31]:
```
aw = {y(t0): y0} # Anfangswerte (initial conditions)
```

[32]:
```
dsolve(dgl, y(t), ics=aw)
```

[32]:
$$y(t) = y_0 e^{t} e^{-t_0}$$

1.3.2 Inhomogene lineare Differentialgleichung erster Ordnung

$$u'(t) = u(t) + \sin(t)$$

[33]:
```
u = Function('u')
t = symbols('t', real=True)
dgl = Eq(u(t).diff(t) - u(t) - sin(t), 0)
sol = dsolve(dgl, u(t))
sol
```

[33]:
$$u(t) = C_1 e^{t} - \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$$

[34]:
```
c1 = solve(sol.subs(t, t0), c1_s)[0]
c1
```

[34]:
$$\left(u(t_0) + \frac{\sqrt{2}\sin\left(t_0 + \frac{\pi}{4}\right)}{2}\right) e^{-t_0}$$

[35]:
```
w = sol.subs(c1_s, c1)
w = w.rhs.subs(t0, 0)
w
```

[35]:
$$\left(u(0) + \frac{1}{2}\right) e^{t} - \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$$

1.3.3 Variation der Konstanten Formel

Die Lösung von

$$\frac{d}{dt}u(t) = au(t) + g(t, u(t)), \quad u(0) = u_0$$

ist

$$u(t) = \mathrm{e}^{at}u_0 + \int_0^t \mathrm{e}^{a(t-\tau)} g(\tau, u(\tau))d\tau$$

Für das Beispiel oben ist $a = 1$ und $g(t, u) = \sin(t)$. Das ergibt:

[36]:
```
v = u(0) * exp(t) + integrate(exp(t - tau) * (sin(tau)), (tau, 0, t))
v
```

[36]:
$$u(0)e^{t} + \frac{e^{t}}{2} - \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$$

[37]:
```
simplify(v - w)
```

[37]:
$$0$$

1.3.4 Lösung mit einem Reihenansatz

$$y'(t) = y(t), \quad y(0) = 1$$

mit einem Reihenansatz

[38]:
```
y0, t, C = symbols('y0 t C')
a = symbols('a:8')
y = Function('y')
```

[39]:
```
dgl = Eq(y(t).diff(t), y(t))
dgl
```

[39]:
$$\frac{d}{dt}y(t) = y(t)$$

[40]:
```
ys = sum([a[i] * t**i for i in range(8)])
ys = ys.subs(a[0], 1)  # y(0) = 1
ys
```

[40]:
$$a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + 1$$

[41]:
```
gl = dgl.subs(y(t), ys).doit()
gl
```

[41]:
$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + 7a_7 t^6 = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + 1$$

[42]:
```
gl.coeff(t) # das klappt so nicht
```

```
---------------------------------------------------------------------------
AttributeError                            Traceback (most recent call last)
Cell In[42], line 1
----> 1 gl.coeff(t) # das klappt so nicht

AttributeError: 'Equality' object has no attribute 'coeff'
```

[43]:
```
gls = gl.as_poly(t).all_coeffs()
gls
```

[43]:
$$[-a_7,\ -a_6 + 7a_7,\ -a_5 + 6a_6,\ -a_4 + 5a_5,\ -a_3 + 4a_4,\ -a_2 + 3a_3,\ -a_1 + 2a_2,\ a_1 - 1]$$

[44]:
```
ac = solve(gls[1:])
ac
```

[44]:
$$\left\{a_1 : 1,\ a_2 : \frac{1}{2},\ a_3 : \frac{1}{6},\ a_4 : \frac{1}{24},\ a_5 : \frac{1}{120},\ a_6 : \frac{1}{720},\ a_7 : \frac{1}{5040}\right\}$$

[45]:
```
ac[a[0]] = 1  # wir raten das Bildungsgesetz der a's
acc = [ac[j] for j in a]
acc
```

[45]:
$$\left[1,\ 1,\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{24},\ \frac{1}{120},\ \frac{1}{720},\ \frac{1}{5040}\right]$$

[46]:
```
[acc[j] / acc[j + 1] for j in range(7)]
```

[46]:
$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7]$$

[47]:
```
[acc[j] - 1 / factorial(j) for j in range(8)]
```

[47]:
$$[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]$$

[48]:
```
n = symbols('n')
yr = Sum(t**n / factorial(n), (n, 0, oo))
yr
```

[48]:
$$\sum_{n=0}^{\infty} \frac{t^n}{n!}$$

[49]:
```
yr.doit()
```

[49]:
$$e^t$$

1.3.5 Logistische Gleichung

einfaches Modell zur Beschreibung von Wachstum in einem Habitat mit beschränkten Resourcen

$$y'(t) = (1 - y(t))y(t)$$

[50]:
```
y = Function('y')
t = symbols('t', real=True)
dgl = Eq(y(t).diff(t), (1 - y(t)) * y(t))
dgl
```

[50]:
$$\frac{d}{dt}y(t) = (1 - y(t))\, y(t)$$

[51]:
```
sol = dsolve(dgl, y(t))
sol
```

[51]:
$$y(t) = \frac{1}{C_1 e^{-t} + 1}$$

[52]:
```
C1 = solve(sol, c1_s)
C1
```

[52]:
$$\left[-e^t + \frac{e^t}{y(t)}\right]$$

[53]:
```
C1 = solve(sol, c1_s).pop().subs(t, t0)
C1
```

[53]:
$$-e^{t_0} + \frac{e^{t_0}}{y(t_0)}$$

[54]:
```
sol.subs(c1_s, C1)
```

[54]:
$$y(t) = \frac{1}{\left(-e^{t_0} + \frac{e^{t_0}}{y(t_0)}\right) e^{-t} + 1}$$