QR Decomposition using Householder Reflections

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Overview

- **Goal:** Decompose an $n \times m$ matrix $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, where:
  - $\boldsymbol{Q}$ is an $n \times n$ orthogonal matrix ($\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}$).
  - $\boldsymbol{R}$ is an $n \times m$ upper triangular matrix.
- Householder reflections are used to construct $\boldsymbol{Q}$.
- Each reflection eliminates elements below the diagonal in a column of $\boldsymbol{A}$.

Step 1: Householder Reflections

- A Householder reflection is represented as:

$$\boldsymbol{H} = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}},$$

  where $\boldsymbol{v}$ is a vector such that $\boldsymbol{H}\boldsymbol{x}$ aligns $\boldsymbol{x}$ with the coordinate axis.
- To eliminate elements below the diagonal in the first column of $\boldsymbol{A}$:
  1. Let $\boldsymbol{x}$ be the first column of $\boldsymbol{A}$.
  2. Define $\boldsymbol{v} = \boldsymbol{x} + \text{sign}(x_1)\|\boldsymbol{x}\|\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the first standard basis vector.
  3. Compute $\boldsymbol{H}_1 = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}$.

Step 2: Iterative Application of Reflections

- Apply $\boldsymbol{H}_1$ to $\boldsymbol{A}$ to zero out sub-diagonal elements in the first column:

$$\boldsymbol{A}_1 = \boldsymbol{H}_1 \boldsymbol{A}.$$

- Repeat for each subsequent column:
  1. Extract the relevant submatrix of $\boldsymbol{A}$.
  2. Construct the Householder reflection $\boldsymbol{H}_k$.
  3. Apply $\boldsymbol{H}_k$ to zero out sub-diagonal elements in the k-th column.
- After m steps, $\boldsymbol{R}$ is upper triangular.

Step 3: Constructing $\boldsymbol{Q}$

- Each Householder matrix $\boldsymbol{H}_k$ is orthogonal.
- The product of all Householder reflections forms $\boldsymbol{Q}^\top$:

$$\boldsymbol{Q}^\top = \boldsymbol{H}_m \boldsymbol{H}_{m-1} \cdots \boldsymbol{H}_1.$$

- Take the transpose to obtain $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \boldsymbol{H}_1 \boldsymbol{H}_2 \cdots \boldsymbol{H}_m.$$

Summary of Algorithm

1. For each column $k = 1, \ldots, m$:
   1.1 Construct the Householder vector $\boldsymbol{v}$.
   1.2 Form $\boldsymbol{H}_k = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}$.
   1.3 Update $\boldsymbol{A} \leftarrow \boldsymbol{H}_k\boldsymbol{A}$.
2. Set $\boldsymbol{R} = \boldsymbol{A}$.
3. Compute $\boldsymbol{Q} = \boldsymbol{H}_1\boldsymbol{H}_2\cdots\boldsymbol{H}_m$.

Properties of QR Decomposition

- $\boldsymbol{Q}$ is orthogonal: $\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}$.
- $\boldsymbol{R}$ is upper triangular.
- Householder reflections ensure numerical stability.