QR Decomposition of an $n \times m$ Matrix

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January 8, 2025

Introduction

- QR decomposition is a method of decomposing a matrix A into:
  $$A = QR$$
  where:
  - Q: an orthogonal (or unitary) matrix.
  - R: an upper triangular matrix.
- Used in solving least squares problems, eigenvalue computations, etc.
- Applicable to any $n \times m$ matrix A where $n \geq m$.

QR Decomposition: Process Overview

1. Start with a matrix A of size $n \times m$.
2. Use either:
   - The Gram-Schmidt process, or
   - Householder reflections.
3. Produce Q, an $n \times n$ orthogonal matrix.
4. Produce R, an $n \times m$ upper triangular matrix.

Gram-Schmidt Process

1. Given matrix $A = [\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_m]$.
2. Initialize $\mathsf{q}_1 = \frac{\mathsf{a}_1}{\|\mathsf{a}_1\|}$.
3. For $j = 2, \ldots, m$:

$$\mathsf{q}_j = \mathsf{a}_j - \sum_{i=1}^{j-1} \mathrm{proj}_{\mathsf{q}_i}(\mathsf{a}_j)$$

   where:

$$\mathrm{proj}_{\mathsf{q}_i}(\mathsf{a}_j) = \frac{\mathsf{q}_i^T \mathsf{a}_j}{\mathsf{q}_i^T \mathsf{q}_i} \mathsf{q}_i.$$

4. Normalize q_j:

$$\mathsf{q}_j = \frac{\mathsf{q}_j}{\|\mathsf{q}_j\|}.$$

5. Form $Q = [\mathsf{q}_1, \mathsf{q}_2, \ldots, \mathsf{q}_m]$.

Householder Reflections

- Use reflections to zero out entries below the diagonal.
- For each column j:
  1. Define a vector v such that:

     $$\mathrm{v} = \mathrm{x} - \|\mathrm{x}\|\mathrm{e}_1$$

     where x is the current column vector.
  2. Construct the Householder matrix:

     $$H = I - 2\frac{\mathrm{v}\mathrm{v}^T}{\mathrm{v}^T\mathrm{v}}.$$

  3. Apply H to zero out entries below the diagonal.
- Repeat for each column.

Example: QR Decomposition

- Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$.
- Apply Gram-Schmidt or Householder method to find:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{3} & \frac{5}{\sqrt{3}} \\ 0 & \sqrt{2} \end{bmatrix}.$$

Applications of QR Decomposition

- Solving least squares problems:

$$\min_{x} \|Ax - b\|.$$

- Eigenvalue computation using QR algorithm.
- Stability in numerical methods.

Thank You! Questions?