lektion12

January 29, 2018

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1 Lektion 12

```
In [1]: from IPython.display import display
        import numpy as np
        import matplotlib.pyplot as plt
        from sympy import *
        init_printing()
        %matplotlib inline
        x,y,z = symbols('x y z')
```

1.1 Daten exportieren und importieren

```
In [2]: H  = Matrix(5,5,[Rational(1,j+i+1) for i in range(5) for j in range(5)])
        H
```

Out[2]:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

```
In [3]: str(H)
```

```
Out[3]: 'Matrix([[1, 1/2, 1/3, 1/4, 1/5], [1/2, 1/3, 1/4, 1/5, 1/6], [1/3, 1/4, 1/5, 1/6, 1/7],
```

```
In [4]: srepr(H) # erzeugt ausfuehrbaren Python code
```

```
Out[4]: 'MutableDenseMatrix([[Integer(1), Rational(1, 2), Rational(1, 3), Rational(1, 4), Ration
```

```
In [5]: preview(H,output='dvi')
```

```
In [6]: with open('output.txt','w') as f:  # w: write
            f.write(srepr(H) + '\n')
```

```
In [7]: %less output.txt
```

```
In [8]: x,y = symbols('x y')
        f = log(x)*sqrt(x**2/3+y**2)
        df = f.diff(x,y).simplify()
        df
        f = Function('f')
        lhs = f(x,y).diff(x,y)
        eq = Eq(lhs, df)
        eq
```

Out[8]:

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\sqrt{3}y}{x\left(x^2+3y^2\right)^{\frac{3}{2}}}\left(-x^2\log\left(x\right)+x^2+3y^2\right)$$

```
In [9]: with open('diff.tex','w') as f:
            f.write(latex(eq) + '\n')
```

```
In [10]: %less diff.tex
```

```
In [11]: z = symbols('z:5')
         V = Matrix(5,5,[z[j]**i for j in range(5) for i in range(5)])
         V
```

Out[11]:

$$\begin{bmatrix}1 & z_0 & z_0^2 & z_0^3 & z_0^4\\ 1 & z_1 & z_1^2 & z_1^3 & z_1^4\\ 1 & z_2 & z_2^2 & z_2^3 & z_2^4\\ 1 & z_3 & z_3^2 & z_3^3 & z_3^4\\ 1 & z_4 & z_4^2 & z_4^3 & z_4^4\end{bmatrix}$$

```
In [12]: with open('output.txt','a') as f:  # 'a' append / anhaengen
             f.write(srepr(2*H) + '\n')
             f.write(srepr(V))
```

```
In [13]: with open('output.txt') as f:  # read (default)
             HH = S(f.readline())
             H2 = S(f.readline())
             VV = S(f.readline())
```

```
In [14]: HH, H2, VV
```

Out[14]:

$$\left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \\ 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \frac{1}{3} & \frac{2}{7} \\ \frac{1}{2} & \frac{2}{5} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} & \frac{2}{9} \end{bmatrix}, \quad \begin{bmatrix} 1 & z_0 & z_0^2 & z_0^3 & z_0^4 \\ 1 & z_1 & z_1^2 & z_1^3 & z_1^4 \\ 1 & z_2 & z_2^2 & z_2^3 & z_2^4 \\ 1 & z_3 & z_3^2 & z_3^3 & z_3^4 \\ 1 & z_4 & z_4^2 & z_4^3 & z_4^4 \end{bmatrix} \right)$$

```
In [15]: %less output.txt
```

```
In [16]: # '%' IPython Magie (shell Befehl 'less')
         # Aendere output.txt in Editor
```

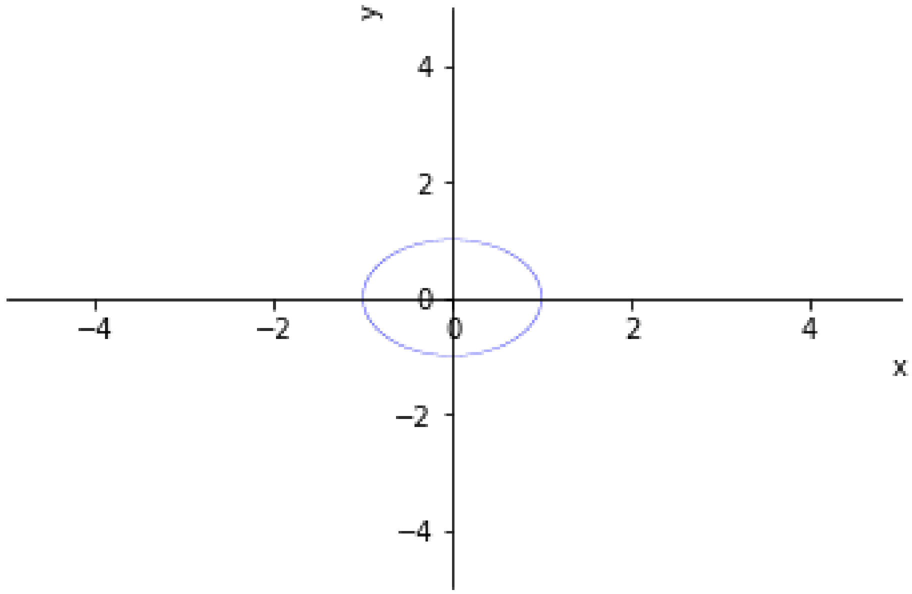
```
In [17]: with open('output.txt') as f:  # read (default)
             for zeile in f:
                 print(zeile)
             else:
                 print('Ende erreicht')
```

```
MutableDenseMatrix([[Integer(1), Rational(1, 2), Rational(1, 3), Rational(1, 4), Rational(1, 5)]

MutableDenseMatrix([[Integer(2), Integer(1), Rational(2, 3), Rational(1, 2), Rational(2, 5)], [I

MutableDenseMatrix([[Integer(1), Symbol('z0'), Pow(Symbol('z0'), Integer(2)), Pow(Symbol('z0'),
Ende erreicht
```

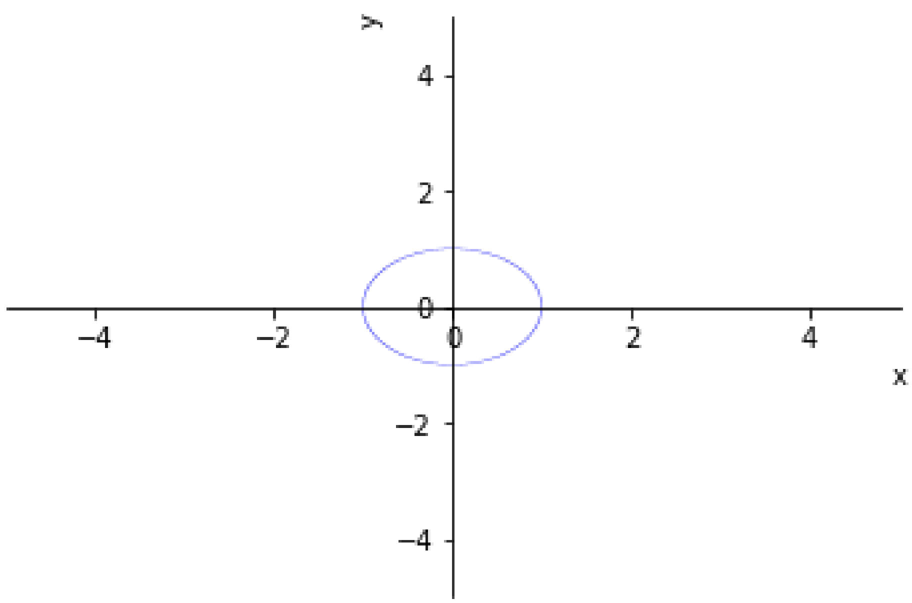
```
In [18]: %less output.txt
```

1.2 Bilder exportieren / speichern

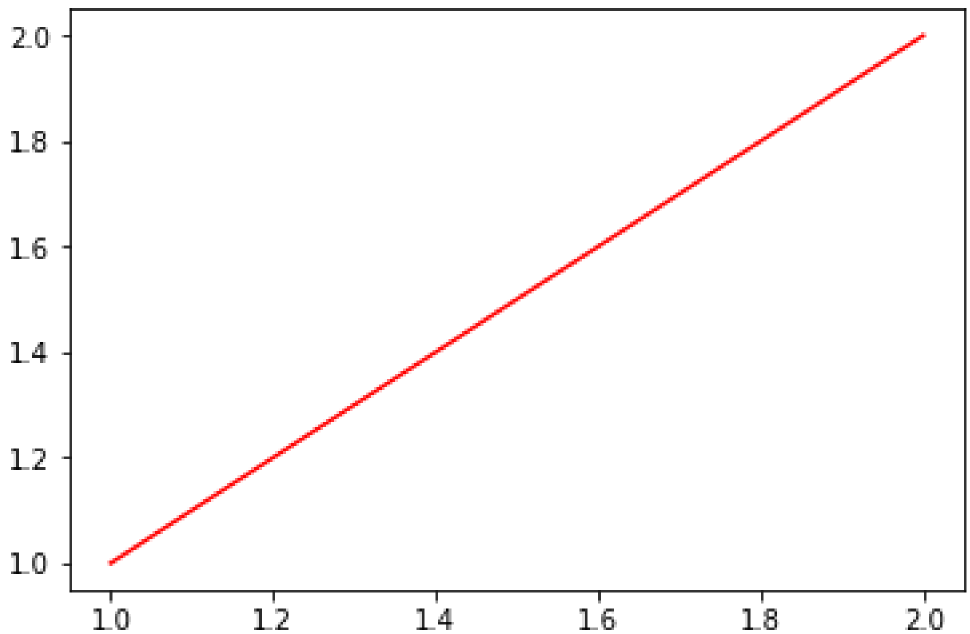
```
In [19]: p = plot_implicit(x**2+y**2-1)
```




```
In [20]: p.save('kreis.png')
```




```
In [21]: fig = plt.figure()
         plt.plot([1,2],[1,2],'r');
```




```
In [22]: fig.savefig('rotelinie.png',format = 'png')
         fig.savefig('rotelinie.pdf',format = 'pdf')
         fig.savefig('rotelinie.eps',format = 'eps')
         fig.savefig('rotelinie.svg',format = 'svg')
```

1.3 Differentialgleichungen

1.3.1 erstes Beispiel

$\dot{y}(t) = y(t)$

```
In [23]: y = Function('y')
         t,tau = symbols('t tau', real = True)
         dgl = Eq(y(t).diff(t)-y(t),0)
         sol = dsolve(dgl,y(t))
         sol
```

Out[23]:

$$y(t) = C_1 e^t$$

```
In [24]: C1 = sol.atoms(Symbol).difference(dgl.atoms(Symbol)).pop()
         C1
```

Out[24]:

$$C_1$$

In [25]:
```
C1 = solve(sol,C1).pop().subs(t,0)
C1
```

Out[25]:

$$y(0)$$

In [26]:
```
checkodesol(dgl,sol)
```

Out[26]: (True, 0)

1.3.2 inhomogene lineare DGL

$$\dot{u}(t) = u(t) + \sin(t)$$

In [27]:
```
u = Function('u')
t = symbols('t', real = True)
dgl = Eq(u(t).diff(t)-u(t)-sin(t))
sol = dsolve(dgl,u(t))
sol
```

Out[27]:

$$u(t) = \left(C_1 - \frac{e^{-t}}{2}\sin(t) - \frac{e^{-t}}{2}\cos(t)\right)e^t$$

In [28]:
```
C1 = sol.atoms(Symbol).difference(dgl.atoms(Symbol)).pop()
C1 = solve(sol,C1).pop().subs(t,0)
C1
```

Out[28]:

$$u(0) + \frac{1}{2}$$

In [29]:
```
w = (sol.subs(sol.atoms(Symbol).difference(dgl.atoms(Symbol)).pop(),C1)).rhs
w
```

Out[29]:

$$\left(u(0) + \frac{1}{2} - \frac{e^{-t}}{2}\sin(t) - \frac{e^{-t}}{2}\cos(t)\right)e^t$$

1.3.3 Variation der Konstanten Formel

In [30]:
```
v = u(0)*exp(t) + integrate(exp(t-tau)*(sin(tau)),(tau,0,t))
v
```

Out[30]:

$$u(0)e^t + \frac{e^t}{2} - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)$$

In [31]:
```
simplify(v-w)
```

Out[31]:

$$0$$

1.3.4 Loesung mit einen Reihenansatz

$$\dot{y}(t) = y(t), y(0) = 1$$

mit einem Reihenansatz

In [32]:
```
y0, t, C = symbols('y0 t C')
a = symbols('a:8')
y = Function('y')
```

In [33]:
```
dgl = Eq(y(t).diff(t),y(t))
dgl
```

Out[33]:

$$\frac{d}{dt}y(t) = y(t)$$

In [34]:
```
ys = sum([a[i]*t**i for i in range(8)])
ys = ys.subs(a[0],1)  # y(0) = 1
ys
```

Out[34]:

$$a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + 1$$

In [35]:
```
gl = dgl.subs(y(t),ys).doit()
gl
```

Out[35]:

$$a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + 7a_7t^6 = a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + 1$$

In [36]:
```
gl.coeff(t)
```

Out[36]:

$$0$$

In [37]:
```
gl1 = (gl.lhs - gl.rhs).expand()
gl1
```

Out[37]:

$$-a_1t + a_1 - a_2t^2 + 2a_2t - a_3t^3 + 3a_3t^2 - a_4t^4 + 4a_4t^3 - a_5t^5 + 5a_5t^4 - a_6t^6 + 6a_6t^5 - a_7t^7 + 7a_7t^6 - 1$$

```
In [38]: gls = gl1.as_poly(t).all_coeffs()
         gls[1:]
```

Out[38]:

$$[-a_6 + 7a_7, \quad -a_5 + 6a_6, \quad -a_4 + 5a_5, \quad -a_3 + 4a_4, \quad -a_2 + 3a_3, \quad -a_1 + 2a_2, \quad a_1 - 1]$$

```
In [39]: ac = solve(gls[1:])
         ac
```

Out[39]:

$$\left\{a_1 : 1, \quad a_2 : \frac{1}{2}, \quad a_3 : \frac{1}{6}, \quad a_4 : \frac{1}{24}, \quad a_5 : \frac{1}{120}, \quad a_6 : \frac{1}{720}, \quad a_7 : \frac{1}{5040}\right\}$$

```
In [40]: ac[a[0]] = 1
         acc = [ac[j]for j in a]
         acc
```

Out[40]:

$$\left[1, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{24}, \quad \frac{1}{120}, \quad \frac{1}{720}, \quad \frac{1}{5040}\right]$$

```
In [41]: [ acc[j]/acc[j+1] for j in range(7)]
```

Out[41]:

$$[1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7]$$

```
In [42]: [ acc[j] - 1/factorial(j) for j in range(8)]
```

Out[42]:

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

```
In [43]: n = symbols('n')
         yr = Sum(t**n/factorial(n) ,(n,0,oo))
         yr
```

Out[43]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}$$

```
In [44]: yr.doit()
```

Out[44]:

$$e^t$$

1.3.5 Logistische Gleichung

$$\dot{y}(t) = (1 - y(t))y(t)$$

```
In [45]: y = Function('y')
         t = symbols('t', real = True)
         dgl = Eq(y(t).diff(t)-(1-y(t))*y(t))
         sol = dsolve(dgl,y(t))
         sol
```

Out[45]:

$$y(t) = -\frac{1}{C_1 e^{-t} - 1}$$

```
In [46]: K1 = sol.atoms(Symbol).difference(dgl.atoms(Symbol)).pop()
         K1
```

Out[46]:

$$C_1$$

```
In [47]: C1 = solve(sol,K1)
         C1
```

Out[47]:

$$\left[e^t - \frac{e^t}{y(t)}\right]$$

```
In [48]: C1 = solve(sol,K1).pop().subs(t,0)
         C1
```

Out[48]:

$$1 - \frac{1}{y(0)}$$

```
In [49]: sol.subs(sol.atoms(Symbol).difference(dgl.atoms(Symbol)).pop(),C1)
```

Out[49]:

$$y(t) = -\frac{1}{\left(1 - \frac{1}{y(0)}\right) e^{-t} - 1}$$

1.3.6 Zweite Ordnung DGL

$$\ddot{y}(t) = -2\dot{y}(t) + y(t) - 100\cos(t)$$

In [50]:
```
dgl2 = Eq(y(t).diff(t,2) + 2*y(t).diff(t) -y(t)+100*cos(t),0)
dgl2
```

Out[50]:

$$-y(t) + 100\cos(t) + 2\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) = 0$$

In [51]:
```
sol = dsolve(dgl2,y(t))
sol
```

Out[51]:

$$y(t) = C_1 e^{t\left(-1+\sqrt{2}\right)} + C_2 e^{t\left(-\sqrt{2}-1\right)} - 25\sin(t) + 25\cos(t)$$

Anfangswerte

$$y(0) = 10,\ \dot{y}(0) = 0$$

In [52]:
```
[Eq(sol.rhs.subs(t,0),10), Eq(sol.rhs.diff(t).subs(t,0),0)]
```

Out[52]:

$$\left[C_1 + C_2 + 25 = 10,\quad C_1\left(-1+\sqrt{2}\right) + C_2\left(-\sqrt{2}-1\right) - 25 = 0\right]$$

In [53]:
```
Constants = solve([Eq(sol.rhs.subs(t,0),10), Eq(sol.rhs.diff(t).subs(t,0),0)])
Constants # Loesung per Hand
```

Out[53]:

$$\left\{C_1 : -\frac{15}{2} + \frac{5\sqrt{2}}{2},\quad C_2 : -\frac{15}{2} - \frac{5\sqrt{2}}{2}\right\}$$

In [54]:
```
# kuerzer
Constants = solve([sol.rhs.subs(t,0)-10, sol.rhs.diff(t).subs(t,0)])
Constants
```

Out[54]:

$$\left\{C_1 : -\frac{15}{2} + \frac{5\sqrt{2}}{2},\quad C_2 : -\frac{15}{2} - \frac{5\sqrt{2}}{2}\right\}$$

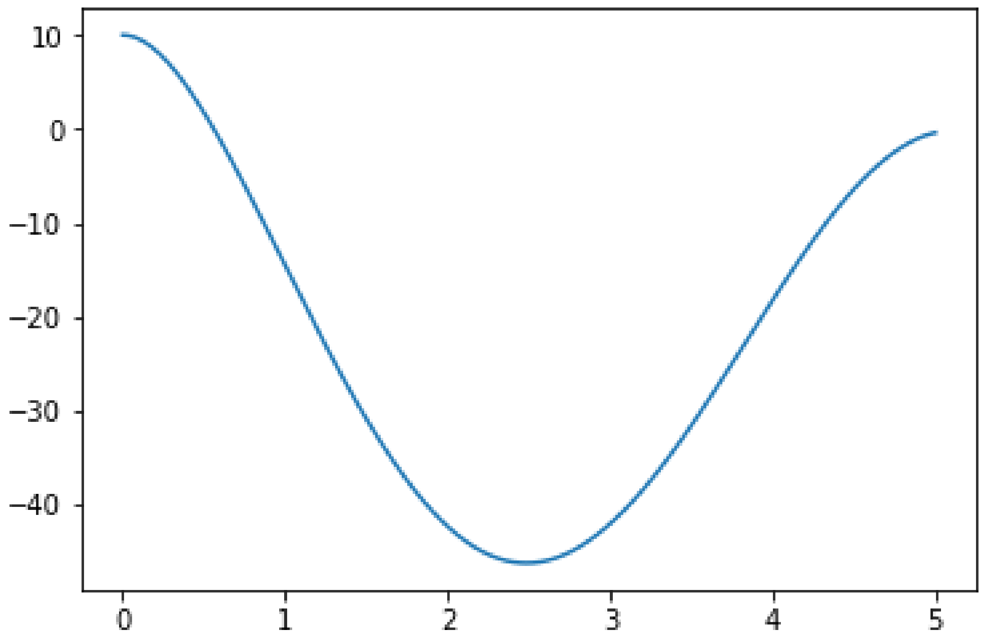
In [55]:
```
sol = sol.subs(Constants)
sol
```

Out[55]:

$$y(t) = \left(-\frac{15}{2} + \frac{5\sqrt{2}}{2}\right) e^{t\left(-1+\sqrt{2}\right)} + \left(-\frac{15}{2} - \frac{5\sqrt{2}}{2}\right) e^{t\left(-\sqrt{2}-1\right)} - 25 \sin\left(t\right) + 25 \cos\left(t\right)$$

```
In [56]: tn = np.linspace(0,5,500)
         yn = lambdify(t,sol.rhs)
         fig = plt.figure()
         plt.plot(tn,yn(tn))
```

```
Out[56]: [<matplotlib.lines.Line2D at 0x7f518cd1bd68>]
```



1.3.7 Matrixexponentialfunktion

```
In [57]: t = symbols('t',real=True)
```

```
In [58]: A = Matrix(3,3,[2,1,0,0,2,1,0,0,2])
         A
```

Out[58]:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

```
In [59]: (t*A).exp()
```

Out[59]:

$$\begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

```
In [60]: u0 = Matrix(3,1,[1,2,3])
         u0
```

Out[60]:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
In [61]: u = (t*A).exp()*u0
         u
```

Out[61]:

$$\begin{bmatrix} \frac{3t^2}{2}e^{2t} + 2te^{2t} + e^{2t} \\ 3te^{2t} + 2e^{2t} \\ 3e^{2t} \end{bmatrix}$$

```
In [62]: diff(u,t) - A*u
```

Out[62]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1.3.8 Gekoppeltes Pendel (Kleinwinkelnaeherung)

$$\ddot{y}(t) = w(t) - y(t) + \cos(2t) \quad (1)$$
$$\ddot{w}(t) = y(t) - w(t) \quad (2)$$

aequivalent zum System erster Ordnung

$$x_0 = y, x_1 = \dot{y}, x_2 = w, x_3 = \dot{w}$$

ergibt

$$\dot{x}_0(t) = x_1(t) \quad (3)$$
$$\dot{x}_1(t) = x_2(t) - x_0(t) + \cos(2t) \quad (4)$$
$$\dot{x}_2(t) = x_3(t) \quad (5)$$
$$\dot{x}_3(t) = x_0(t) - x_2(t) \quad (6)$$

```
In [63]: y = Function('y')
         w = Function('w')
         t, tau = symbols('t tau',real=True)
```

```
In [64]: dgl = (Eq( y(t).diff(t,2), w(t)-y(t)+cos(t)), \
         Eq( w(t).diff(t,2), y(t)-w(t)))
         dgl
```

Out[64]:

$$\left(\frac{d^{2}}{dt^{2}} y(t) = w(t) - y(t) + \cos(t), \quad \frac{d^{2}}{dt^{2}} w(t) = -w(t) + y(t)\right)$$

```
In [65]: #dsolve(dgl,(y(t),w(t)))
```

```
In [66]: A = Matrix(4,4,[0,1,0,0, -1,0,1,0, 0,0,0,1, 1,0,-1,0])
         A
```

Out[66]:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

```
In [67]: Tt = (t*A).exp()
         Tt
```

Out[67]:

$$\begin{bmatrix} \frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} + \frac{1}{4}e^{-\sqrt{2}it} & \frac{t}{2} - \frac{\sqrt{2}i}{8}e^{\sqrt{2}it} + \frac{\sqrt{2}i}{8}e^{-\sqrt{2}it} & -\frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} - \frac{1}{4}e^{-\sqrt{2}it} & \frac{t}{2} + \frac{\sqrt{2}i}{8}e^{\sqrt{2}it} - \frac{\sqrt{2}i}{8}e^{-\sqrt{2}it} \\ \frac{\sqrt{2}i}{4}e^{\sqrt{2}it} - \frac{\sqrt{2}i}{4}e^{-\sqrt{2}it} & \frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} + \frac{1}{4}e^{-\sqrt{2}it} & -\frac{\sqrt{2}i}{4}e^{\sqrt{2}it} + \frac{\sqrt{2}i}{4}e^{-\sqrt{2}it} & -\frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} - \frac{1}{4}e^{-\sqrt{2}it} \\ -\frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} - \frac{1}{4}e^{-\sqrt{2}it} & \frac{t}{2} + \frac{\sqrt{2}i}{8}e^{\sqrt{2}it} - \frac{\sqrt{2}i}{8}e^{-\sqrt{2}it} & \frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} + \frac{1}{4}e^{-\sqrt{2}it} & \frac{t}{2} - \frac{\sqrt{2}i}{8}e^{\sqrt{2}it} + \frac{\sqrt{2}i}{8}e^{-\sqrt{2}it} \\ -\frac{\sqrt{2}i}{4}e^{\sqrt{2}it} + \frac{\sqrt{2}i}{4}e^{-\sqrt{2}it} & -\frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} - \frac{1}{4}e^{-\sqrt{2}it} & \frac{\sqrt{2}i}{4}e^{\sqrt{2}it} - \frac{\sqrt{2}i}{4}e^{-\sqrt{2}it} & \frac{1}{4}e^{\sqrt{2}it} + \frac{1}{2} + \frac{1}{4}e^{-\sqrt{2}it} \end{bmatrix}$$

```
In [68]: Tttau = Matrix(4,4,[x.rewrite(sin).expand().subs(t,t-tau) for x in Tt])
         Tttau
```

Out[68]:

$$\begin{bmatrix} \frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & \frac{t}{2} - \frac{\tau}{2} + \frac{\sqrt{2}}{4}\sin\left(\sqrt{2}(t-\tau)\right) & -\frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & \frac{t}{2} - \frac{\tau}{2} - \frac{\sqrt{2}}{4}\sin\left(\sqrt{2}(t-\tau)\right) \\ -\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}(t-\tau)\right) & \frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & \frac{\sqrt{2}}{2}\sin\left(\sqrt{2}(t-\tau)\right) & -\frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} \\ -\frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & \frac{t}{2} - \frac{\tau}{2} - \frac{\sqrt{2}}{4}\sin\left(\sqrt{2}(t-\tau)\right) & \frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & \frac{t}{2} - \frac{\tau}{2} + \frac{\sqrt{2}}{4}\sin\left(\sqrt{2}(t-\tau)\right) \\ \frac{\sqrt{2}}{2}\sin\left(\sqrt{2}(t-\tau)\right) & -\frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} & -\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}(t-\tau)\right) & \frac{1}{2}\cos\left(\sqrt{2}(t-\tau)\right) + \frac{1}{2} \end{bmatrix}$$

```
In [69]: Ttt = Matrix(4,4,[x.rewrite(sin).expand() for x in Tt])
         Ttt
```

Out[69]:

$$\left[\begin{matrix}\frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & \frac{t}{2}+\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right) & -\frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & \frac{t}{2}-\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)\\-\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}t\right) & \frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & \frac{\sqrt{2}}{2}\sin\left(\sqrt{2}t\right) & -\frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2}\\-\frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & \frac{t}{2}-\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right) & \frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & \frac{t}{2}+\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)\\\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}t\right) & -\frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2} & -\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}t\right) & \frac{1}{2}\cos\left(\sqrt{2}t\right)+\frac{1}{2}\end{matrix}\right]$$

In [70]:
```
u0 = Matrix(4,1,[1,0,1,0])
ftau  = Matrix(4,1,[0,0,cos(2*tau),0])
ft  = Matrix(4,1,[0,0,cos(2*t),0])
u0,ftau, ft
```

Out[70]:

$$\left(\left[\begin{matrix}1\\0\\1\\0\end{matrix}\right], \left[\begin{matrix}0\\0\\\cos\left(2\tau\right)\\0\end{matrix}\right], \left[\begin{matrix}0\\0\\\cos\left(2t\right)\\0\end{matrix}\right]\right)$$

In [71]:
```
u = Ttt*u0 + integrate(Tttau*ftau,(tau,0,t))
```

In [72]:
```
u = u.doit()
u
```

Out[72]:

$$\left[\begin{matrix}-\frac{1}{4}\sin\left(2t\right)+\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)+1\\\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\cos\left(2t\right)+\frac{\sqrt{2}}{2}\cos\left(\sqrt{2}t\right)\right)\\\frac{3}{4}\sin\left(2t\right)-\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)+1\\-\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\cos\left(2t\right)+\frac{\sqrt{2}}{2}\cos\left(\sqrt{2}t\right)\right)\end{matrix}\right]$$

In [73]:
```
simplify(u.diff(t) - A*u - ft)
```

Out[73]:

$$\left[\begin{matrix}0\\0\\0\\0\end{matrix}\right]$$

In [74]:
```
A = Matrix(2,2,[2,1,0,1])
A
```

Out[74]:

$$\left[\begin{matrix}2 & 1\\0 & 1\end{matrix}\right]$$

```
In [75]: A.exp()
```

Out[75]:

$$\begin{bmatrix} e^2 & -e+e^2 \\ 0 & e \end{bmatrix}$$

1.3.9 Pendelgleichung 2. Versuch

$$\ddot{y}(t) = w(t) - y(t) + \cos(2t) \tag{7}$$

$$\ddot{w}(t) = y(t) - \mathbf{3}w(t) \tag{8}$$

Die Loesung von

$$\ddot{x} = -Bx + g$$

ist gegeben durch

$$x(t) = C_1 \cos(t\sqrt{B}) + C_2\sqrt{B}^{-1}\sin(t\sqrt{B}) + \int_0^t \sqrt{B}^{-1}\sin((t-\tau)\sqrt{B})g(\tau)d\tau$$

```
In [76]: B = Matrix(2,2,[1,-1,-1,3])
         T, D = B.diagonalize()
         B,D,T, simplify(B- T*D*T.inv())
```

Out[76]:

$$\left(\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{2}+2 & 0 \\ 0 & \sqrt{2}+2 \end{bmatrix}, \quad \begin{bmatrix} 1+\sqrt{2} & -\sqrt{2}+1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

```
In [77]: CosBt = simplify(T*diag(cos(t*sqrt(D[0,0])),cos(t*sqrt(D[1,1])))*T.inv())
         SinBttau = simplify(T*diag(sin((t-tau)*sqrt(D[0,0]))/sqrt(D[0,0]),\
                                    sin((t-tau)*sqrt(D[1,1]))/sqrt(D[1,1]))*T.inv())

         v0 = Matrix(2,1,[1,2])
         gtau = Matrix(2,1,[cos(2*tau),0])
         gt   = Matrix(2,1,[cos(2*t),0])

         display(SinBttau, CosBt, v0, gtau, gt)
```

$$\begin{bmatrix} \frac{1}{4}\left(1+\sqrt{2}\right)\sqrt{\sqrt{2}+2}\sin\left(\sqrt{-\sqrt{2}+2}\,(t-\tau)\right) - \frac{1}{4}\left(-\sqrt{2}+1\right)\sqrt{-\sqrt{2}+2}\sin\left(\sqrt{\sqrt{2}+2}\,(t-\tau)\right) & \\ \frac{1}{4}\sqrt{\sqrt{2}+2}\sin\left(\sqrt{-\sqrt{2}+2}\,(t-\tau)\right) - \frac{1}{4}\sqrt{-\sqrt{2}+2}\sin\left(\sqrt{\sqrt{2}+2}\,(t-\tau)\right) & -\frac{1}{4}\left(-\sqrt{2}\right. \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{4}\left(\left(1+\sqrt{2}\right)\cos\left(t\sqrt{-\sqrt{2}+2}\right) + \left(-1+\sqrt{2}\right)\cos\left(t\sqrt{\sqrt{2}+2}\right)\right) & \frac{\sqrt{2}}{4}\left(\cos\left(t\sqrt{-\sqrt{2}+2}\right) - \cos\left(\right.\right. \\ \frac{\sqrt{2}}{4}\left(\cos\left(t\sqrt{-\sqrt{2}+2}\right) - \cos\left(t\sqrt{\sqrt{2}+2}\right)\right) & \frac{\sqrt{2}}{4}\left(\left(-1+\sqrt{2}\right)\cos\left(t\sqrt{-\sqrt{2}+2}\right) + \left(1+\right.\right. \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix}\cos(2\tau)\\0\end{bmatrix}$$

$$\begin{bmatrix}\cos(2t)\\0\end{bmatrix}$$

In [78]:
```
v = CosBt*v0 + integrate(SinBttau*gtau,(tau,0,t))
v = v.doit()
v
```

Out[78]:

$$\begin{bmatrix}\frac{\sqrt{2}}{4}\left(\left(1+\sqrt{2}\right)\cos\left(t\sqrt{-\sqrt{2}+2}\right)+\left(-1+\sqrt{2}\right)\cos\left(t\sqrt{\sqrt{2}+2}\right)\right)+\frac{\sqrt{2}}{2}\left(\cos\left(t\sqrt{-\sqrt{2}+2}\right)-\cos\left(t\sqrt{\sqrt{2}+\right.\right.\\ \frac{\sqrt{2}}{2}\left(\left(-1+\sqrt{2}\right)\cos\left(t\sqrt{-\sqrt{2}+2}\right)+\left(1+\sqrt{2}\right)\cos\left(t\sqrt{\sqrt{2}+2}\right)\right)+\frac{\sqrt{2}}{4}\left(\cos\left(t\sqrt{-\sqrt{2}+2}\right)-\cos\left(t\sqrt{\sqrt{2}+2}\right)\right.\end{bmatrix}$$

In [79]:
```
v.simplify()
```

Out[79]:

$$\begin{bmatrix}-\frac{1}{2}\cos(2t)+\frac{3}{4}\cos\left(t\sqrt{-\sqrt{2}+2}\right)+\frac{3\sqrt{2}}{4}\cos\left(t\sqrt{-\sqrt{2}+2}\right)-\frac{3\sqrt{2}}{4}\cos\left(t\sqrt{\sqrt{2}+2}\right)+\frac{3}{4}\cos\left(t\sqrt{\sqrt{2}+2}\right)\\ \frac{1}{2}\cos(2t)+\frac{3}{4}\cos\left(t\sqrt{-\sqrt{2}+2}\right)+\frac{3}{4}\cos\left(t\sqrt{\sqrt{2}+2}\right)\end{bmatrix}$$

In [80]:
```
simplify(v.diff(t,2) + B*v - gt ) , v.subs(t,0)-v0
```

Out[80]:

$$\left(\begin{bmatrix}0\\0\end{bmatrix},\quad\begin{bmatrix}0\\0\end{bmatrix}\right)$$