Aufgabe 55

```
> restart:
> with(LinearAlgebra):
> with(VectorCalculus):
> BasisFormat(false):
```

```
> A := < < 1, 0, 0, 0 > | < 1, 1, 0, 0 > | < 0, 1, 1, 0 > | < 0,
  0, 0, 2 > >;
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

```
> y0 := < 1, 1, 1, 1 >;
```

$$y0 := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.1}$$

```
> # Löse u1'(x) = A u(x), u(0) = y0.
> u1 := x -> MatrixExponential(A, x) . y0:
> u1(x);
```

$$\begin{bmatrix} e^x + x e^x + \frac{1}{2} x^2 e^x \\ e^x + x e^x \\ e^x \\ e^{2x} \end{bmatrix} \tag{1.2}$$

```
> # Probe:
> diff(u1(x), x) - A . u1(x);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.3}$$

```
> # Inhomogene Gleichung
> g := x -> < sin(x), 0, x, 0 >;
```

$$g := x \mapsto \langle \sin(x), 0, x, 0 \rangle \tag{1.4}$$

```
> # Variation-der-Konstanten-Formel:
> u2 := x -> u1(x) + int(MatrixExponential(A, x - s) . g(s), s =
  0 .. x);
```

$$u2 := x \mapsto u1(x) + \int_0^x LinearAlgebra:-MatrixExponential(A, x + (-s)) \cdot g(s)\, ds \tag{1.5}$$

```
> # Probe
> diff(u2(x), x) - (A . u2(x) + g(x));
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.6}$$

```
> # Variante mit dsolve. Benötigt Maple-Version >= 18
> y := x -> < y1(x), y2(x), y3(x), y4(x) >;
```

$$y := x \mapsto \langle y1(x), y2(x), y3(x), y4(x) \rangle \tag{1.7}$$

```
> dgl := { diff(y(x), x) - A . y(x) };
> aw := { y(0) - y0 };
```

$$dgl := \left\{ \begin{bmatrix} \frac{d}{dx} y1(x) - y1(x) - y2(x) \\ \frac{d}{dx} y2(x) - y2(x) - y3(x) \\ \frac{d}{dx} y3(x) - y3(x) \\ \frac{d}{dx} y4(x) - 2\, y4(x) \end{bmatrix} \right\}$$

$$aw := \left\{ \begin{bmatrix} y1(0) - 1 \\ y2(0) - 1 \\ y3(0) - 1 \\ y4(0) - 1 \end{bmatrix} \right\} \tag{1.8}$$

```
> v1 := x -> rhs(dsolve(dgl union aw, y(x)));
> v1(x);
```

$$v1 := x \mapsto rhs(dsolve(dgl \cup aw, y(x)))$$

Error, (in dsolve) invalid arguments: expected an equation, or a set or list of them, received: {Vector(4, {(1) = diff(y1(x), x)-y1(x)-y2(x), (2) = diff(y2(x), x)-y2(x)-y3(x), (3) = diff(y3(x), x)-y3(x), (4) = diff(y4(x), x)-2*y4(x)}, attributes = [coords = cartesian])}

```
> # Inhomogene Gleichung
> dgl2 := { diff(y(x), x) - A . y(x) - g(x) };
```

$$dgl2 := \left\{ \left[\begin{array}{c} \frac{d}{dx} y1(x) - y1(x) - y2(x) - \sin(x) \\ \frac{d}{dx} y2(x) - y2(x) - y3(x) \\ \frac{d}{dx} y3(x) - y3(x) - x \\ \frac{d}{dx} y4(x) - 2\, y4(x) \end{array} \right] \right\} \quad (1.9)$$

```
> v2 := x -> rhs(dsolve(dgl2 union aw, y(x)));
> v2(x);
```

$$v2 := x \mapsto rhs(dsolve(dgl2 \cup aw, y(x)))$$

```
Error, (in dsolve) invalid arguments; expected an equation,
or a set or list of them, received: {Vector(4, {(1) = diff(y1
(x), x)-y1(x)-y2(x)-sin(x), (2) = diff(y2(x), x)-y2(x)-y3(x),
(3) = diff(y3(x), x)-y3(x)-x, (4) = diff(y4(x), x)-2*y4(x)},
attributes = [coords = cartesian])}
> # Und natürlich wieder prüfen (s.o.) ...
```

Aufgabe 56

```
> restart:
> # Nun etwas allgemeiner, als in A 55.
> dgl := { diff(y(x), x) = A * y(x) + f(x, y(x)) };
> aw := { y(0) = y0 };
```

$$dgl := \{y'(x) = A\, y(x) + f(x, y(x))\}$$

$$aw := \{y(0) = y0\} \quad (2.1)$$

```
> Phi := exp(x * A) * y0 + int(exp((x - s) * A) * f(s, y(s)), s =
  0 .. x);
```

$$\Phi := e^{xA}\, y0 + \int_0^x e^{(x-s)\,A}\, f(s, y(s))\, ds \quad (2.2)$$

```
> # Probe
> simplify(diff(Phi, x) - (A * Phi + f(x, y(x))));
```

$$0 \quad (2.3)$$

Aufgabe 57

```
> restart:
> with(plots):
> dgl := {
    diff(y(x), x) = a * y(x)  - b * y(x) * z(x),
    diff(z(x), x) = -c * z(x) + d * y(x) * z(x)
  };
> aw := { y(0) = y0, z(0) = z0 };
```

$$dgl := \{y'(x) = a\,y(x) - b\,y(x)\,z(x),\ z'(x) = -c\,z(x) + d\,y(x)\,z(x)\}$$

$$aw := \{y(0) = y0,\ z(0) = z0\} \qquad (3.1)$$

```
> params := { y0 = 6000, z0 = 30, a = 1/5, b = 1/500, c = 1/10, d
  = 1/100000 };
> dgl2 := { op(subs(params, dgl)), op(subs(params, aw)) };
```

$$params := \left\{a = \frac{1}{5},\ b = \frac{1}{500},\ c = \frac{1}{10},\ d = \frac{1}{100000},\ y0 = 6000,\ z0 = 30\right\}$$

$$dgl2 := \left\{y'(x) = \frac{y(x)}{5} - \frac{y(x)\,z(x)}{500},\ z'(x) = -\frac{z(x)}{10} + \frac{y(x)\,z(x)}{100000},\ y(0) = 6000,\ z(0) = 30\right\} \qquad (3.2)$$

```
> loesung := dsolve(dgl2, { y(x), z(x) }, numeric, output =
  listprocedure);
```

$$loesung := [x = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc},\ y(x) = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc},\ z(x) = \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc}] \qquad (3.3)$$

```
> loesung(1);
```

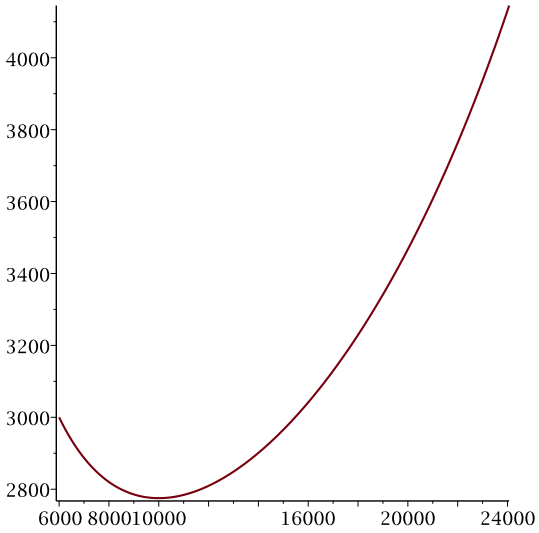
$$[x(1) = 1.,\ y(x)(1) = 6909.23308579663,\ z(x)(1) = 28.9516039379840] \qquad (3.4)$$

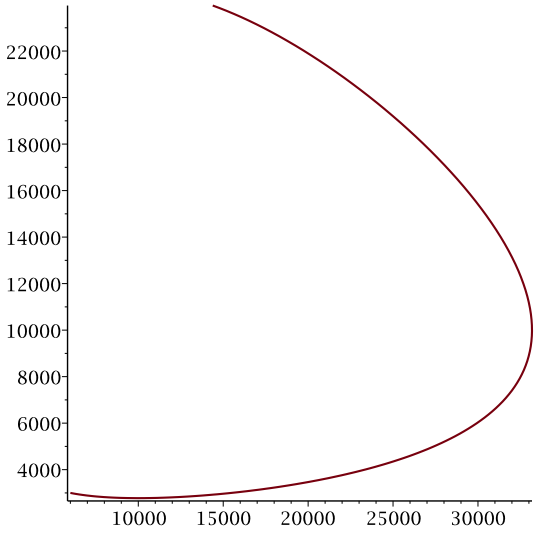
```
> ly := x -> rhs(loesung[2](x));
> lz := x -> rhs(loesung[3](x));
```

$$ly := x \mapsto rhs(loesung_2(x))$$

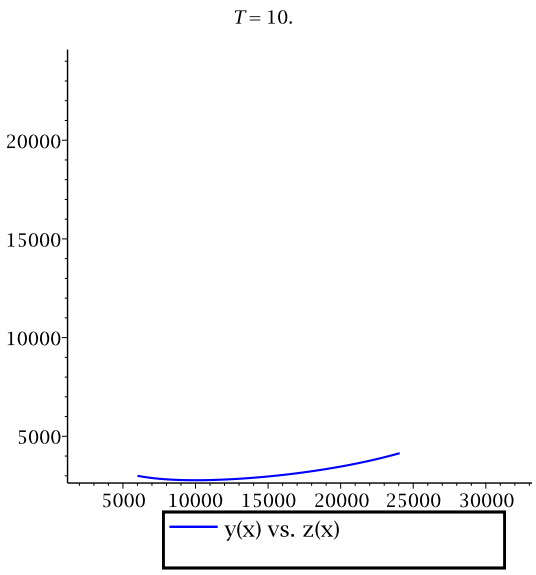
$$lz := x \mapsto rhs(loesung_3(x)) \qquad (3.5)$$

```
> # Suche Plotbereich
> plot([ ly(x), 100 * lz(x), x = 0 .. 10]);
> plot([ ly(x), 100 * lz(x), x = 0 .. 20]);
```


4000
3800
3600
3400
3200
3000
2800
6000 800010000
16000
20000
24000




```
> animate(plot, [ [ ly(x), 100 * lz(x), x = T - 10 .. T ], color
  = blue, legend = "y(x) vs. z(x)" ], T = 10 .. 60);
```



(b)

```
> eqs := seq(subs({ y(x) = y, z(x) = z }, rhs(dgl[kk]) = 0), kk =
  1..nops(dgl));
```

$$eqs := -b\,y\,z + a\,y = 0,\ d\,y\,z - c\,z = 0 \qquad \textbf{(3.6)}$$

```
> gleichGewPunkte := solve({ eqs }, { y, z });
```

$$gleichGewPunkte := \{y = 0,\ z = 0\},\ \left\{y = \frac{c}{d},\ z = \frac{a}{b}\right\} \qquad \textbf{(3.7)}$$

```
> # Probe:
> for ggp in gleichGewPunkte do
    awGG := subs(subs({ y = y0, z = z0 }, ggp), aw);
    dsolve({ op(dgl), op(awGG) }, { y(x), z(x) });
  end do;
```

$$awGG := \{y(0) = 0,\ z(0) = 0\}$$

$$\{y(x) = 0,\ z(x) = 0\}$$

$$awGG := \left\{y(0) = \frac{c}{d},\ z(0) = \frac{a}{b}\right\} \qquad \textbf{(3.8)}$$

$$\left\{y(x) = \frac{c}{d},\ z(x) = \frac{a}{b}\right\} \tag{3.8}$$

```
> # Also Lösungen konstant!
```

(c)

```
> # Werte aus (a)
> # Richtungsfeld der DGL
> dgl2;
```

$$\left\{y'(x) = \frac{y(x)}{5} - \frac{y(x)\,z(x)}{500},\ z'(x) = -\frac{z(x)}{10} + \frac{y(x)\,z(x)}{100000},\ y(0) = 6000,\ z(0) = 30\right\} \tag{3.9}$$

```
> v := <seq(subs({ y(x) = y, z(x) = z }, rhs(dgl2[kk])), kk = 1 .
  . nops(dgl)) >;
```

$$v := \begin{bmatrix} \frac{1}{5}\,y - \frac{1}{500}\,yz \\ -\frac{1}{10}\,z + \frac{1}{100000}\,yz \end{bmatrix} \tag{3.10}$$

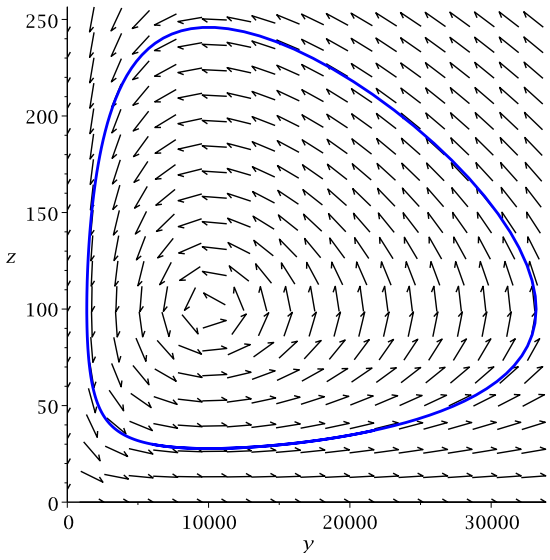
```
> # Vektorfeld normieren
> w := v / norm(v, 2);
```

$$w := \begin{bmatrix} \frac{\frac{1}{5}\,y - \frac{1}{500}\,yz}{\sqrt{\left|-\frac{1}{5}\,y + \frac{1}{500}\,yz\right|^2 + \left|-\frac{1}{10}\,z + \frac{1}{100000}\,yz\right|^2}} \\ \frac{-\frac{1}{10}\,z + \frac{1}{100000}\,yz}{\sqrt{\left|-\frac{1}{5}\,y + \frac{1}{500}\,yz\right|^2 + \left|-\frac{1}{10}\,z + \frac{1}{100000}\,yz\right|^2}} \end{bmatrix} \tag{3.11}$$

```
> pf := fieldplot(w, y = 0..33000, z = 0 .. 250);
> psol := plot([ ly(x), lz(x), x = 0 .. 60 ], color = blue,
  thickness = 2);
> display(pf, psol);
```

$$pf := PLOT(...)$$

$$psol := PLOT(...)$$



Aufgabe 58

```
> restart:
> with(plots):
> dgl := {
    diff(y(x), x) = a * y(x) - k*y(x)^2 - b * y(x) * z(x),
    diff(z(x), x) = -c * z(x) + d * y(x) * z(x)
  };
```

$$dgl := \{y'(x) = a\,y(x) - k\,y(x)^2 - b\,y(x)\,z(x),\ z'(x) = -c\,z(x) + d\,y(x)\,z(x)\} \quad \textbf{(4.1)}$$

```
> aw := { y(0) = y0, z(0) = 0 };
```

$$aw := \{y(0) = y0,\ z(0) = 0\} \quad \textbf{(4.2)}$$

```
> loes := dsolve({ op(dgl), op(aw) }, { y(x), z(x) });
```

$$loes := \left\{y(x) = \frac{a}{\frac{e^{-ax}\,(-k\,y0 + a)}{y0} + k},\ z(x) = 0\right\} \quad \textbf{(4.3)}$$

```
> limit(rhs(loes[1]), x = infinity) assuming a::positive;
```

(4.4)

$$\frac{a}{k} \tag{4.4}$$

(c)

```
> params := { y0 = 6000, z0 = 30, a = 1/5, b = 1/500, c = 1/10, d
  = 1/100000 };
```

$$params := \left\{a = \frac{1}{5}, b = \frac{1}{500}, c = \frac{1}{10}, d = \frac{1}{100000}, y0 = 6000, z0 = 30\right\} \tag{4.5}$$

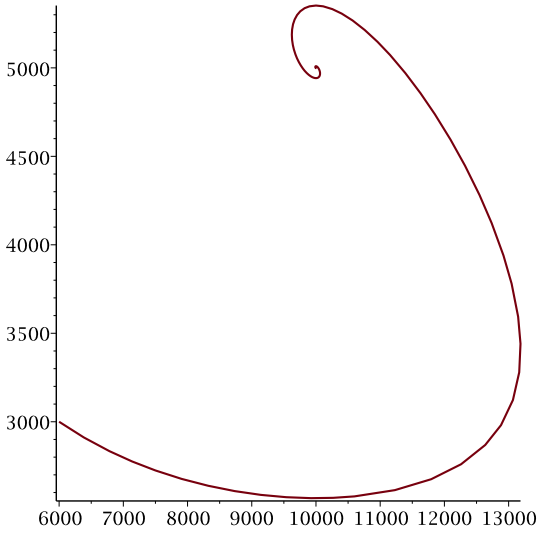
```
> ks := { k = 1/10^5, k = 1/10^6 };
```

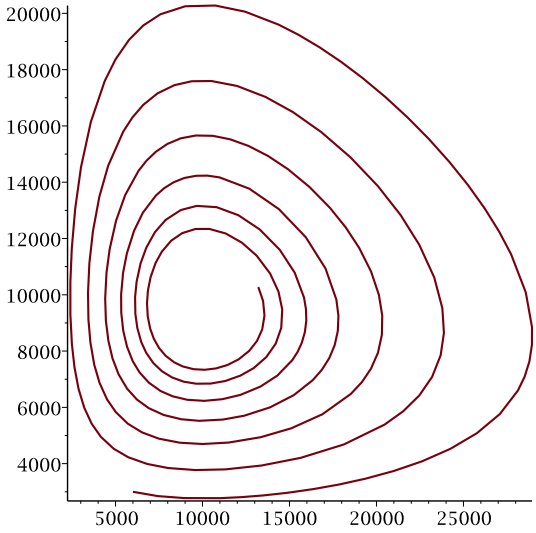
$$ks := \left\{k = \frac{1}{100000}, k = \frac{1}{1000000}\right\} \tag{4.6}$$

```
> aw2 := { y(0) = y0, z(0) = z0 };
```

$$aw2 := \{y(0) = y0, z(0) = z0\} \tag{4.7}$$

```
> for kk in 1 .. nops(ks) do
    dgl2 := subs(params union {ks[kk]}, dgl) union subs(params,
  aw2):
    loesung := dsolve(dgl2, { y(x), z(x) }, numeric, output =
  listprocedure):
    ly := x -> rhs(loesung[2](x));
    lz := x -> rhs(loesung[3](x));
    print(plot([ ly(x), 100 * lz(x), x = 0 .. 300]));
  end do:
```


5000
4500
4000
3500
3000
6000
7000
8000
9000
10000
11000
12000
13000



Aufgabe 59

```
> restart:
> with(plots):
> with(VectorCalculus):
> with(LinearAlgebra):
> BasisFormat(false):
```

```
> f := (x, y) -> x^2 * (y + 1) + y/2;
> g := (x, y) -> x^2 + y^2 - 1;
```

$$f := (x, y) \mapsto x^2 (y+1) + \left(y \cdot \left(\frac{1}{2}\right)\right)$$

$$g := (x, y) \mapsto x^2 + y^2 - 1 \qquad \textbf{(5.1)}$$

```
> param := t -> ( cos(t), sin(t) );
```

$$param := t \mapsto (\cos(t), \sin(t)) \qquad \textbf{(5.2)}$$

```
> h := t -> (f@param)(t);
> dh := D(h);
> d2h := D(dh);
```

$$h := t \mapsto (f@param)(t)$$

$$dh := t \mapsto -2\cos(t)\,(\sin(t)+1)\sin(t) + \cos(t)^3 + \frac{\cos(t)}{2}$$

$$d2h := t \mapsto 2\sin(t)^2\,(\sin(t)+1) - 5\cos(t)^2\sin(t) - 2\cos(t)^2\,(\sin(t)+1) \quad \textbf{(5.3)}$$

$$\_\ \frac{\sin(t)}{2}$$

```
> # h(0), dh(0), d2h(0);
> kritische_punkte := solve({ dh(t) = 0 }, { t }):
> # Und wie beim letzten Mal ist die Periodizität bei t = Pi/2
  anders, daher t = -Pi/2 hinzufügen
> kritische_punkte := kritische_punkte, { t = -Pi/2 };
```

$$kritische\_punkte := \left\{t = \arctan\left(\frac{6\left(-\frac{1}{3}+\frac{\sqrt{22}}{6}\right)}{\sqrt{10+4\sqrt{22}}}\right)\right\}, \left\{t = -\arctan\left(\frac{6\left(-\frac{1}{3}+\frac{\sqrt{22}}{6}\right)}{\sqrt{10+4\sqrt{22}}}\right)+\pi\right\}, \left\{t = \arctan\left(-\frac{1}{3}-\frac{\sqrt{22}}{6}, \frac{\sqrt{10-4\sqrt{22}}}{6}\right)\right\}, \left\{t = \arctan\left(-\frac{1}{3}-\frac{\sqrt{22}}{6}, -\frac{\sqrt{10-4\sqrt{22}}}{6}\right)\right\}, \left\{t = \frac{\pi}{2}\right\}, \left\{t = -\frac{\pi}{2}\right\} \quad \textbf{(5.4)}$$

```
> for kr in kritische_punkte do
    xy := simplify(subs(kr, [param(t)])):
    print('t' = simplify(subs(kr, t)));
    print('<x, y>' = xy);
    if (not is(xy[1], real)) or
       (not is(xy[2], real)) then
     # Komplexe Werte überspringen
      print("Komplexe kritische Stelle ignoriert");
      next;
    end if;
    print('f(x, y)' = f(xy[1], xy[2]));
   # Kriterium 2. Ordnung
   d2h_val := simplify(subs(kr, d2h(t)));
   print('diff(f@param, t)'('t') = d2h_val);
   #typ := minMax[sign(d2h_val)];
   #print(typ);
  end do:
```

$$t = \arctan\left(\frac{-2+\sqrt{22}}{\sqrt{10+4\sqrt{22}}}\right)$$

$$\langle x, y\rangle = \left[\frac{\sqrt{2}\,\sqrt{5+2\sqrt{2}\,\sqrt{11}}}{6}, -\frac{1}{3}+\frac{\sqrt{2}\,\sqrt{11}}{6}\right]$$

$$f(x, y) = \left(\frac{5}{18}+\frac{\sqrt{2}\,\sqrt{11}}{9}\right)\left(\frac{2}{3}+\frac{\sqrt{2}\,\sqrt{11}}{6}\right)-\frac{1}{6}+\frac{\sqrt{2}\,\sqrt{11}}{12}$$

$$\left(\frac{\partial}{\partial t}\,(f@param)\right)(t) = -\frac{\left(22\sqrt{2}+5\sqrt{11}\right)\sqrt{2}}{18}$$

$$t = -\arctan\left(\frac{-2+\sqrt{22}}{\sqrt{10+4\sqrt{22}}}\right)+\pi$$

$$\langle x, y\rangle = \left[-\frac{\sqrt{2}\,\sqrt{5+2\sqrt{2}\,\sqrt{11}}}{6}, -\frac{1}{3}+\frac{\sqrt{2}\,\sqrt{11}}{6}\right]$$

$$f(x, y) = \left(\frac{5}{18}+\frac{\sqrt{2}\,\sqrt{11}}{9}\right)\left(\frac{2}{3}+\frac{\sqrt{2}\,\sqrt{11}}{6}\right)-\frac{1}{6}+\frac{\sqrt{2}\,\sqrt{11}}{12}$$

$$\left(\frac{\partial}{\partial t}\,(f@param)\right)(t) = -\frac{\left(22\sqrt{2}+5\sqrt{11}\right)\sqrt{2}}{18}$$

$$t = \arctan\left(-\frac{1}{3}-\frac{\sqrt{22}}{6}, \frac{\mathrm{I}}{6}\sqrt{-10+4\sqrt{22}}\right)$$

$$\langle x, y\rangle = \left[\frac{\mathrm{I}}{6}\sqrt{-10+4\sqrt{22}}, -\frac{1}{3}-\frac{\sqrt{22}}{6}\right]$$

"Komplexe kritische Stelle ignoriert"

$$t = \arctan\left(-\frac{1}{3}-\frac{\sqrt{22}}{6}, -\frac{\mathrm{I}}{6}\sqrt{-10+4\sqrt{22}}\right)$$

$$\langle x, y\rangle = \left[-\frac{\mathrm{I}}{6}\sqrt{-10+4\sqrt{22}}, -\frac{1}{3}-\frac{\sqrt{22}}{6}\right]$$

"Komplexe kritische Stelle ignoriert"

$$t = \frac{\pi}{2}$$

$$\langle x, y\rangle = [0, 1]$$

$$f(x, y) = \frac{1}{2}$$

$$\left(\frac{\partial}{\partial t}\,(f@param)\right)(t) = \frac{7}{2}$$

$$t = -\frac{\pi}{2}$$

$$\langle x, y\rangle = [0, -1]$$

$$f(x, y) = -\frac{1}{2}$$

$$\left(\frac{\partial}{\partial t}\,(f@param)\right)(t) = \frac{1}{2} \tag{5.5}$$

```
> kritische_punkte := seq(kritische_punkte[ll], ll in [ 1, 2, 5,
  6 ]);
```

$$kritische\_punkte := \left\{t = \arctan\left(\frac{6\left(-\frac{1}{3}+\frac{\sqrt{22}}{6}\right)}{\sqrt{10+4\sqrt{22}}}\right)\right\}, \left\{t = -\arctan\left(\frac{6\left(-\frac{1}{3}+\frac{\sqrt{22}}{6}\right)}{\sqrt{10+4\sqrt{22}}}\right)+\pi\right\}, \left\{t = \frac{\pi}{2}\right\}, \left\{t = -\frac{\pi}{2}\right\} \tag{5.6}$$

```
> # Jetzt das Innere betrachten.
> gradF := Gradient(f(x, y), [ x, y]);
```

$$gradF := \begin{bmatrix} 2x(y+1) \\ x^2+\frac{1}{2} \end{bmatrix} \tag{5.7}$$

```
> kritische_punkte_innen := solve({ gradF[1] = 0, gradF[2] = 0 },
  { x, y });
```

$$kritische\_punkte\_innen := \{x = RootOf(2\_Z^2+1), y = -1\} \tag{5.8}$$

```
> for kr in allvalues(kritische_punkte_innen) do
    #xy := simplify(subs(kr, [param(t)])):
    xy := subs(kr, < x, y >);
    print('<x, y>' = xy);
    if (not is(xy[1], real)) or
       (not is(xy[2], real)) then
     # Komplexe Werte überspringen
      print("Komplexe kritische Stelle ignoriert");
       next;
    end if;
  end do:
```

$$\langle x, y\rangle = \begin{bmatrix} \frac{1}{2}\,I\sqrt{2} \\ -1 \end{bmatrix}$$

"Komplexe kritische Stelle ignoriert"

$$\langle x, y\rangle = \begin{bmatrix} -\frac{1}{2}\,I\sqrt{2} \\ -1 \end{bmatrix}$$

"Komplexe kritische Stelle ignoriert" (5.9)

```
> # Also keine kritischen Punkte gefunden, also keine Extrema im
  Inneren.
```

```
> kp3d := [ seq(subs(kr, [ param(t), (f@param)(t) ]), kr in
  kritische_punkte) ]:
> pf := plot3d(f(x, y), x = -1.1..1.1, y = -1.1..1.1):
> pf_constr := spacecurve([ param(t)[1], param(t)[2], (f@param)
  (t), t = -Pi..Pi ], x = -1..1, y = -1..1, thickness = 3, color
  = black ):
> pp := pointplot3d(kp3d, symbol = circle,  symbolsize = 50,
  color = yellow);
> display([ pf, pf_constr, pp ]);
```

$$pp := PLOT3D(\ldots)$$