

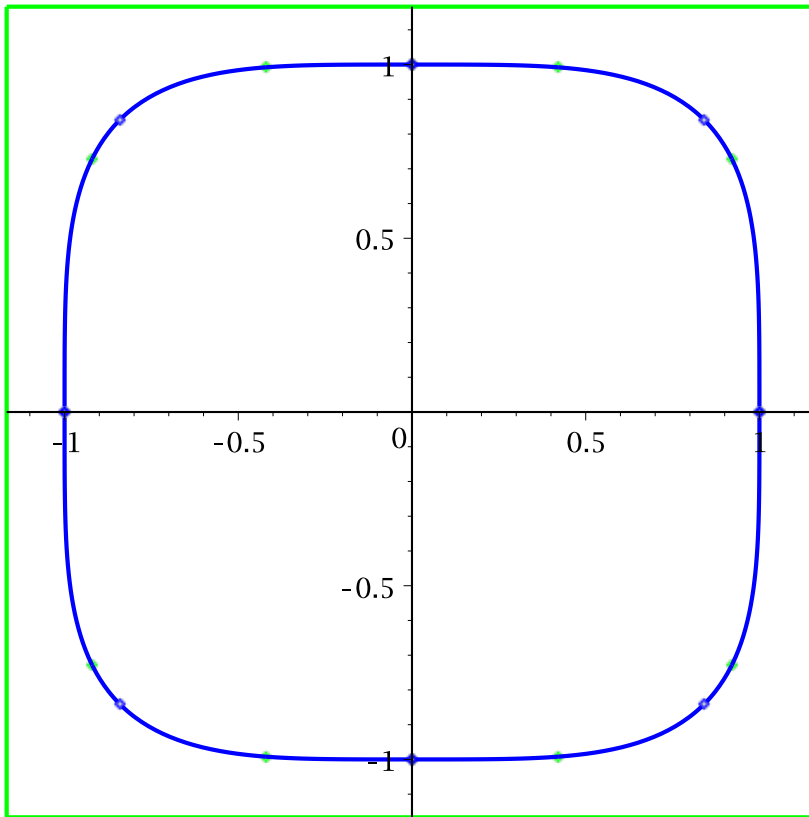
# Computergestuetzte Mathematik zur Analysis

## Lektion 12 (16. Januar)

```
[> restart:
```

### Extrema unter Nebenbedingungen

```
[> with(plots):  
[> with(algcurves):  
[> with(VectorCalculus):  
[> BasisFormat(false):  
[> g := x^4 + y^4 - 1;  
                                      $g := x^4 + y^4 - 1$  (1.1)  
[> NB := plot_real_curve(g, x, y);  
                                      $NB := PLOT(\dots)$  (1.2)  
[> display(NB, scaling = constrained, thickness = 2);
```



```
> f := x + y;
```

$$f := x + y$$

(1.3)

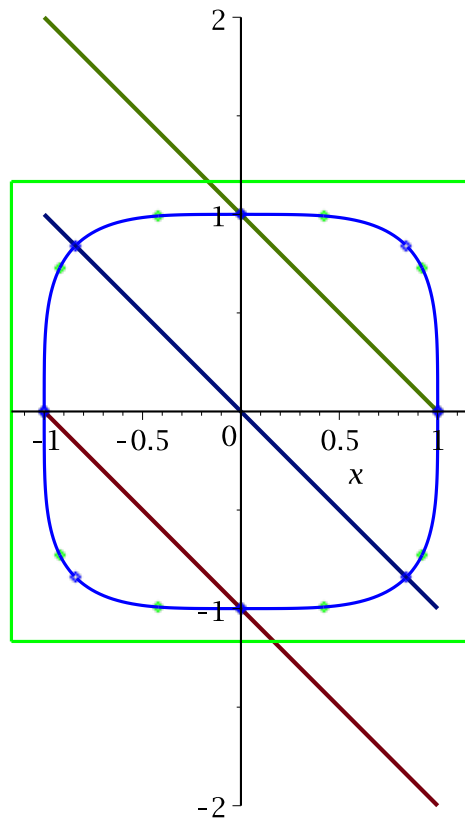
```
> Nf := seq(solve(f = k, y), k=-1..1);
```

$$Nf := -1 - x, -x, 1 - x$$

(1.4)

```
> pl_NF := plot([Nf], x = -1 .. 1, thickness = 2): # 1. Frage
```

```
> display({NB, pl_NF}, scaling = constrained);
```



```
> grad_g := Gradient(g, [x, y]);
```

$$\text{grad}_g := \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix} \quad (1.5)$$

```
> grad_f := Gradient(f, [x, y]);
```

$$\text{grad}_f := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (1.6)$$

```
> GLF := grad_f - lambda*grad_g; # Gradient Lagrangefunktion
```

$$\text{GLF} := \begin{bmatrix} -4\lambda x^3 + 1 \\ -4\lambda y^3 + 1 \end{bmatrix} \quad (1.7)$$

```
> M:= {solve({GLF[1] = 0, GLF[2] = 0, g = 0}), {x, y, lambda}};
```

$$M := \left\{ \left\{ \lambda, x, y \right\}, \left\{ \lambda = -\frac{1}{4} \left( \text{RootOf}(-Z^2 - Z + 1) - 1 \right) \text{RootOf}(-Z^4 - \text{RootOf}(-Z^2 - Z + 1)), x = \text{RootOf}(-Z^4 - \text{RootOf}(-Z^2 - Z + 1)), y = \right. \right. \quad (1.8)$$

$$- \text{RootOf}(\_Z^4 - \text{RootOf}(\_Z^2 - \_Z + 1)) \text{RootOf}(\_Z^2 - \_Z + 1) \}, \left\{ \lambda \right. \\ \left. = \frac{1}{2} \text{RootOf}(2 \_Z^4 - 1), x = \text{RootOf}(2 \_Z^4 - 1), y = \text{RootOf}(2 \_Z^4 - 1) \right\}$$

> M1 := M[1];

$$M1 := \{\lambda, x, y\} \quad (1.9)$$

> M2 := M[2];

$$M2 := \left\{ \lambda = -\frac{1}{4} (\text{RootOf}(\_Z^2 - \_Z + 1) - 1) \text{RootOf}(\_Z^4 - \text{RootOf}(\_Z^2 - \_Z + 1)) \right. \\ \left. + 1), x = \text{RootOf}(\_Z^4 - \text{RootOf}(\_Z^2 - \_Z + 1)), y = -\text{RootOf}(\_Z^4 - \right. \\ \left. - \text{RootOf}(\_Z^2 - \_Z + 1)) \text{RootOf}(\_Z^2 - \_Z + 1) \right\} \quad (1.10)$$

> subs(M2, x);

$$\text{RootOf}(\_Z^4 - \text{RootOf}(\_Z^2 - \_Z + 1)) \quad (1.11)$$

> allvalues((1.11));

$$\frac{1}{2} \sqrt{2I+2\sqrt{3}}, -\frac{1}{2} \sqrt{-2I-2\sqrt{3}}, -\frac{1}{2} \sqrt{2I+2\sqrt{3}}, \frac{1}{2} \sqrt{-2I-2\sqrt{3}}, \\ \frac{1}{2} \sqrt{-2I+2\sqrt{3}}, \frac{1}{2} \sqrt{2I-2\sqrt{3}}, -\frac{1}{2} \sqrt{-2I+2\sqrt{3}}, -\frac{1}{2} \sqrt{2I-2\sqrt{3}} \quad (1.12)$$

> simplify(evalc([(1.12)]));

$$\left[ \frac{1}{4} \sqrt{2} \sqrt{3} + \frac{1}{4} \sqrt{2} + \frac{1}{4} I \sqrt{2} \sqrt{3} - \frac{1}{4} I \sqrt{2}, -\frac{1}{4} \sqrt{2} \sqrt{3} + \frac{1}{4} \sqrt{2} \right. \\ \left. + \frac{1}{4} I \sqrt{2} \sqrt{3} + \frac{1}{4} I \sqrt{2}, -\frac{1}{4} \sqrt{2} \sqrt{3} - \frac{1}{4} \sqrt{2} - \frac{1}{4} I \sqrt{2} \sqrt{3} + \frac{1}{4} I \sqrt{2}, \right. \\ \left. \frac{1}{4} \sqrt{2} \sqrt{3} - \frac{1}{4} \sqrt{2} - \frac{1}{4} I \sqrt{2} \sqrt{3} - \frac{1}{4} I \sqrt{2}, \frac{1}{4} \sqrt{2} \sqrt{3} + \frac{1}{4} \sqrt{2} \right. \\ \left. - \frac{1}{4} I \sqrt{2} \sqrt{3} + \frac{1}{4} I \sqrt{2}, \frac{1}{4} \sqrt{2} \sqrt{3} - \frac{1}{4} \sqrt{2} + \frac{1}{4} I \sqrt{2} \sqrt{3} + \frac{1}{4} I \sqrt{2}, \right. \\ \left. -\frac{1}{4} \sqrt{2} \sqrt{3} - \frac{1}{4} \sqrt{2} + \frac{1}{4} I \sqrt{2} \sqrt{3} - \frac{1}{4} I \sqrt{2}, -\frac{1}{4} \sqrt{2} \sqrt{3} + \frac{1}{4} \sqrt{2} \right. \\ \left. - \frac{1}{4} I \sqrt{2} \sqrt{3} - \frac{1}{4} I \sqrt{2} \right] \quad (1.13)$$

Alle diese Loesungen haben einen nicht-verschwindenden Imaginaerteil. Zur Kontrolle

> evalf(%, 2);

$$[0.95 + 0.25 I, -0.25 + 0.95 I, -0.95 - 0.25 I, 0.25 - 0.95 I, 0.95 - 0.25 I, 0.25 \\ + 0.95 I, -0.95 + 0.25 I, -0.25 - 0.95 I] \quad (1.14)$$

> M3 := M[3];

$$(1.15)$$

$$M3 := \left\{ \lambda = \frac{1}{2} \operatorname{RootOf}(2\_Z^4 - 1), x = \operatorname{RootOf}(2\_Z^4 - 1), y = \operatorname{RootOf}(2\_Z^4 - 1) \right\} \quad (1.15)$$

```
> allvalues(subs(M3, x));
```

$$\frac{1}{2} 2^{3/4}, \frac{1}{2} I 2^{3/4}, -\frac{1}{2} 2^{3/4}, -\frac{1}{2} I 2^{3/4} \quad (1.16)$$

```
> select(t -> not has(t, I), {%});
```

$$\left\{ -\frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4} \right\} \quad (1.17)$$

```
> x1 := %[1]; x2 := %[2];
```

$$x1 := -\frac{1}{2} 2^{3/4}$$

$$x2 := \frac{1}{2} 2^{3/4} \quad (1.18)$$

```
> allvalues(subs(M3, y));
```

$$\frac{1}{2} 2^{3/4}, \frac{1}{2} I 2^{3/4}, -\frac{1}{2} 2^{3/4}, -\frac{1}{2} I 2^{3/4} \quad (1.19)$$

```
> select(t -> not has(t, I), {%});
```

$$\left\{ -\frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4} \right\} \quad (1.20)$$

```
> y1 := %[1]; y2 := %[2];
```

$$y1 := -\frac{1}{2} 2^{3/4}$$

$$y2 := \frac{1}{2} 2^{3/4} \quad (1.21)$$

```
> allvalues(subs(M3, lambda));
```

$$\frac{1}{4} 2^{3/4}, \frac{1}{4} I 2^{3/4}, -\frac{1}{4} 2^{3/4}, -\frac{1}{4} I 2^{3/4} \quad (1.22)$$

```
> select(t -> not has(t, I), {%});
```

$$\left\{ -\frac{1}{4} 2^{3/4}, \frac{1}{4} 2^{3/4} \right\} \quad (1.23)$$

```
> lambda1 := %[1]; lambda2 := %[2]; # 2. Frage
```

$$\lambda1 := -\frac{1}{4} 2^{3/4}$$

$$\lambda2 := \frac{1}{4} 2^{3/4} \quad (1.24)$$

```
> it := 1; for x in [x1, x2] do;
>   for y in [y1, y2] do;
>     for lambda in [lambda1, lambda2] do;
>       print(it, x, y, lambda, GLF); it := it + 1;
```

```

> od;
> od;
> od;
> x := 'x': y := 'y': lambda := 'lambda':

```

$$1, -\frac{1}{2} 2^{3/4}, -\frac{1}{2} 2^{3/4}, -\frac{1}{4} 2^{3/4}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2, -\frac{1}{2} 2^{3/4}, -\frac{1}{2} 2^{3/4}, \frac{1}{4} 2^{3/4}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$3, -\frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4}, -\frac{1}{4} 2^{3/4}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$4, -\frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4}, \frac{1}{4} 2^{3/4}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$5, \frac{1}{2} 2^{3/4}, -\frac{1}{2} 2^{3/4}, -\frac{1}{4} 2^{3/4}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$6, \frac{1}{2} 2^{3/4}, -\frac{1}{2} 2^{3/4}, \frac{1}{4} 2^{3/4}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$7, \frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4}, -\frac{1}{4} 2^{3/4}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$8, \frac{1}{2} 2^{3/4}, \frac{1}{2} 2^{3/4}, \frac{1}{4} 2^{3/4}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(1.25)

```

> punkt1 := {x = x1, y = y1, lambda = lambda1};

```

$$punkt1 := \left\{ \lambda = -\frac{1}{4} 2^{3/4}, x = -\frac{1}{2} 2^{3/4}, y = -\frac{1}{2} 2^{3/4} \right\}$$

(1.26)

```

> punkt2 := {x = x2, y = y2, lambda = lambda2};

```

$$punkt2 := \left\{ \lambda = \frac{1}{4} 2^{3/4}, x = \frac{1}{2} 2^{3/4}, y = \frac{1}{2} 2^{3/4} \right\}$$

(1.27)

Test

```

> subs(punkt1, g);

```

0

(1.28)

```
> subs(punkt2, g);
```

$$0 \quad (1.29)$$

```
> H := Hessian[VectorCalculus](f-lambda*g,[x,y]);
```

$$H := \begin{bmatrix} -12\lambda x^2 & 0 \\ 0 & -12\lambda y^2 \end{bmatrix} \quad (1.30)$$

```
> subs(punkt1,H);
```

$$\begin{bmatrix} 3 \cdot 2^{1/4} & 0 \\ 0 & 3 \cdot 2^{1/4} \end{bmatrix} \quad (1.31)$$

Also haben wir hier ein Minimum (Achtung: Fuer das Minimum muss nur ein Teil der Hessematrix positiv definit sein

-> Optimierung

```
> subs(punkt2,H);
```

$$\begin{bmatrix} -3 \cdot 2^{1/4} & 0 \\ 0 & -3 \cdot 2^{1/4} \end{bmatrix} \quad (1.32)$$

Und hier ein Maximum

```
> wert := subs(punkt1, f);
```

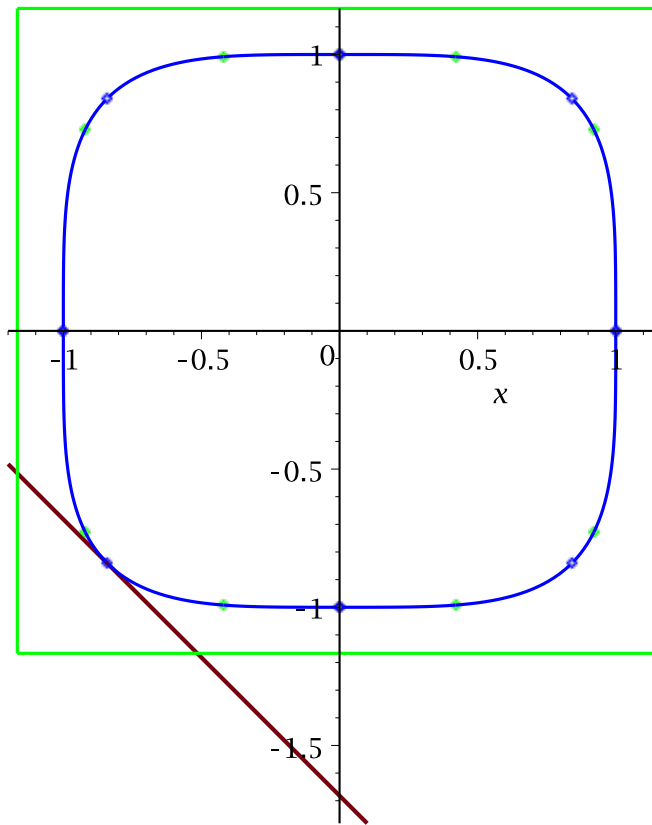
$$\text{wert} := -2^{3/4} \quad (1.33)$$

```
> solve(f = wert, y);
```

$$-2^{3/4} - x \quad (1.34)$$

```
> pl_NFs := plot(%, x = -1.2 .. .1, thickness = 2):
```

```
> display({NB, pl_NFs}, scaling = constrained);
```



## ▼ Extrema unter Nebenbedingungen in 3D

```
> f := x^2 + y^2 + z^2; # 3. Frage
      f := x2 + y2 + z2 (2.1)
```

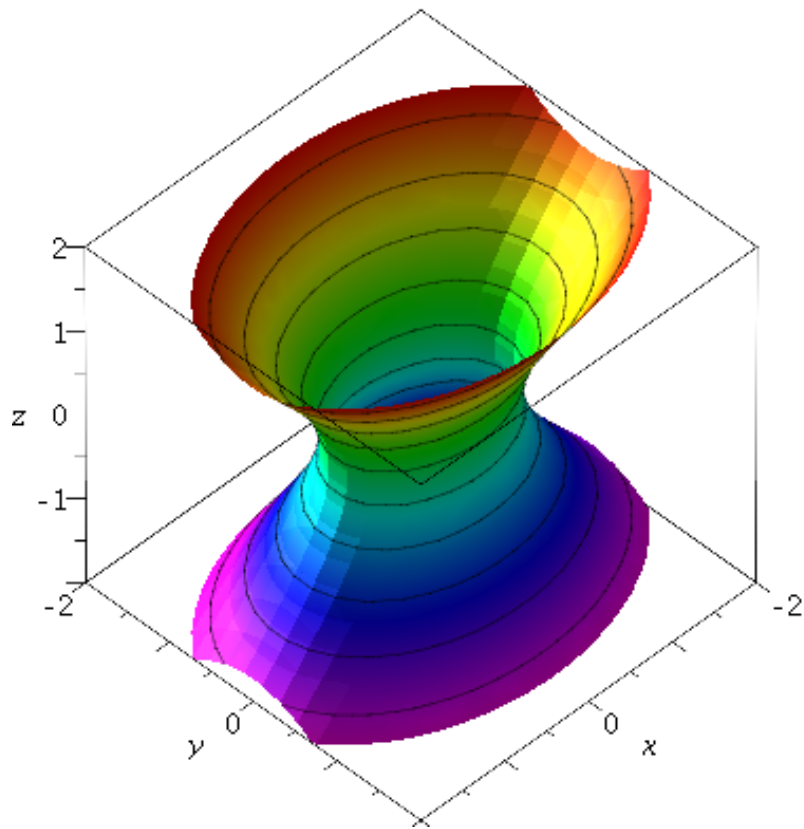
```
> g := x^2 + 2*y^2 - z^2 - 1;
      g := x2 + 2y2 - z2 - 1 (2.2)
```

```
> with(plots):
```

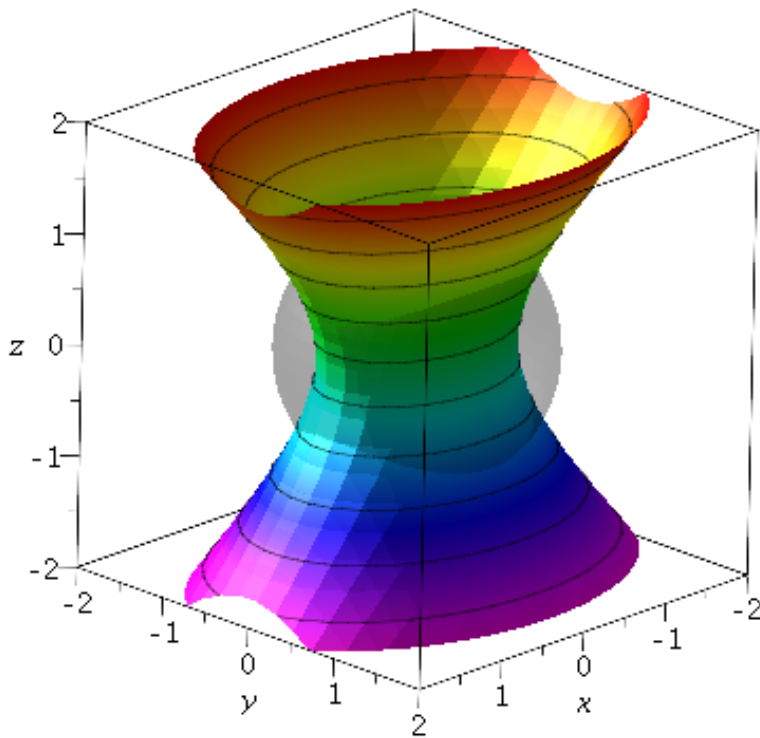
```
> gp := implicitplot3d(g, x=-2..2, y=-2..2, z=-2..2, style =
  patchcontour, shading = zhue, scaling = constrained, axes =
  boxed, numpoints = 5000):
```

```
> gp;
```





```
> Nf3o2 := implicitplot3d(f=1.5, x = -1.5 .. 1.5, y = -1.5 ..  
1.5, z = -1.5..1.5, style = patchnogrid, color = black,  
transparency = .8,numpoints=6000):  
> display({Nf3o2, gp},scaling = constrained);
```



```
> grad_f := Gradient(f, [x,y,z]);
```

$$\text{grad}_f := \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

(2.3)

```
> grad_g := Gradient(g, [x, y, z]);
```

$$\text{grad}_g := \begin{bmatrix} 2x \\ 4y \\ -2z \end{bmatrix}$$

(2.4)

```
> GLF := grad_f - lambda*grad_g;
```

$$\text{GLF} := \begin{bmatrix} -2\lambda x + 2x \\ -4\lambda y + 2y \\ 2\lambda z + 2z \end{bmatrix}$$

(2.5)

```
> Lsg := solve({GLF[1] = 0, GLF[2] = 0, GLF[3] = 0, g = 0}, {x,
```

```
y, z, lambda});
```

$$\text{Lsg} := \{\lambda = 1, x = 1, y = 0, z = 0\}, \{\lambda = 1, x = -1, y = 0, z = 0\}, \left\{\lambda = \frac{1}{2}, x = 0, y = \text{RootOf}(2\_Z^2 - 1), z = 0\right\}, \{\lambda = -1, x = 0, y = 0, z = \text{RootOf}(\_Z^2 + 1)\} \quad (2.6)$$

```
> allvalues(Lsg[4]);
```

$$\{\lambda = -1, x = 0, y = 0, z = 1\}, \{\lambda = -1, x = 0, y = 0, z = -1\} \quad (2.7)$$

Loesungen sind imaginær

```
> G1 := Lsg[1]; G2 := Lsg[2]; G3:=allvalues(Lsg[3]);
```

$$G1 := \{\lambda = 1, x = 1, y = 0, z = 0\}$$

$$G2 := \{\lambda = 1, x = -1, y = 0, z = 0\}$$

$$G3 := \left\{\lambda = \frac{1}{2}, x = 0, y = \frac{1}{2}\sqrt{2}, z = 0\right\}, \left\{\lambda = \frac{1}{2}, x = 0, y = -\frac{1}{2}\sqrt{2}, z = 0\right\} \quad (2.8)$$

```
> f1 := subs(G1, f); g1:=subs(G1,g);
```

$$f1 := 1$$

$$g1 := 0 \quad (2.9)$$

```
> f2 := subs(G2, f); g2:=subs(G2,g);
```

$$f2 := 1$$

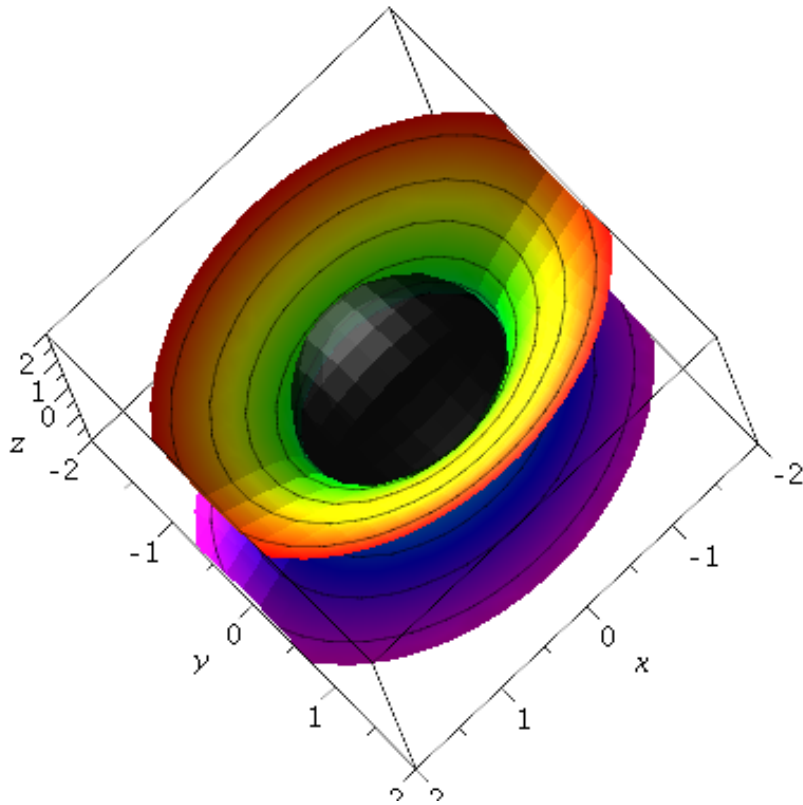
$$g2 := 0 \quad (2.10)$$

```
> wert12 := f1;
```

$$\text{wert12} := 1 \quad (2.11)$$

```
> ball1 := implicitplot3d(f-wert12, x = -1 .. 1, y = -1..1, z=-1..1, style = patchnogrid, color = black, transparency=0.2);
```

```
> display({gp, ball1}, scaling = constrained);
```



```
> f3:=subs(G3[1],f);
```

$$f3 := \frac{1}{2} \quad (2.12)$$

```
> f4:=subs(G3[2],f);
```

$$f4 := \frac{1}{2} \quad (2.13)$$

```
> ball2 := implicitplot3d(f-1/2, x = -1 .. 1, y = -1..1, z=-1..1,
style = patchngrid, color = black, transparency=0.2):
```

```
> display({gp, ball2}, scaling = constrained, orientation = [-30,
35]);
```



```

> for j from 2 to 5 do;
>   Lsg := solve(x^j = 1, x):
>   print(Lsg[j-1]);
> od:

```

$$\begin{array}{c}
 1 \\
 -\frac{1}{2} - \frac{1}{2} I\sqrt{3} \\
 | \\
 -\frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4} I\sqrt{2}\sqrt{5-\sqrt{5}}
 \end{array}
 \tag{3.6}$$

```

> for j from 2 to 5 do;
>   Lsg := solve(x^j = 1, x):
>   print(cat(j,"te Einheitswurzel:"), (Lsg[j-1]));
> od:

```

$$\begin{array}{l}
 \text{2te Einheitswurzel; } 1 \\
 \text{3te Einheitswurzel; } -\frac{1}{2} - \frac{1}{2} I\sqrt{3} \\
 \text{4te Einheitswurzel; } I \\
 \text{5te Einheitswurzel; } -\frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4} I\sqrt{2}\sqrt{5-\sqrt{5}}
 \end{array}
 \tag{3.7}$$

```

> f := x^2 - 1/2;

```

$$f := x^2 - \frac{1}{2}
 \tag{3.8}$$

```

> convert(f, string);

```

$$\text{"x^2-1/2"}
 \tag{3.9}$$

```

> Beschreibung := "Graph von " || (convert(f, string));
Beschreibung := "Graph von x^2-1/2"

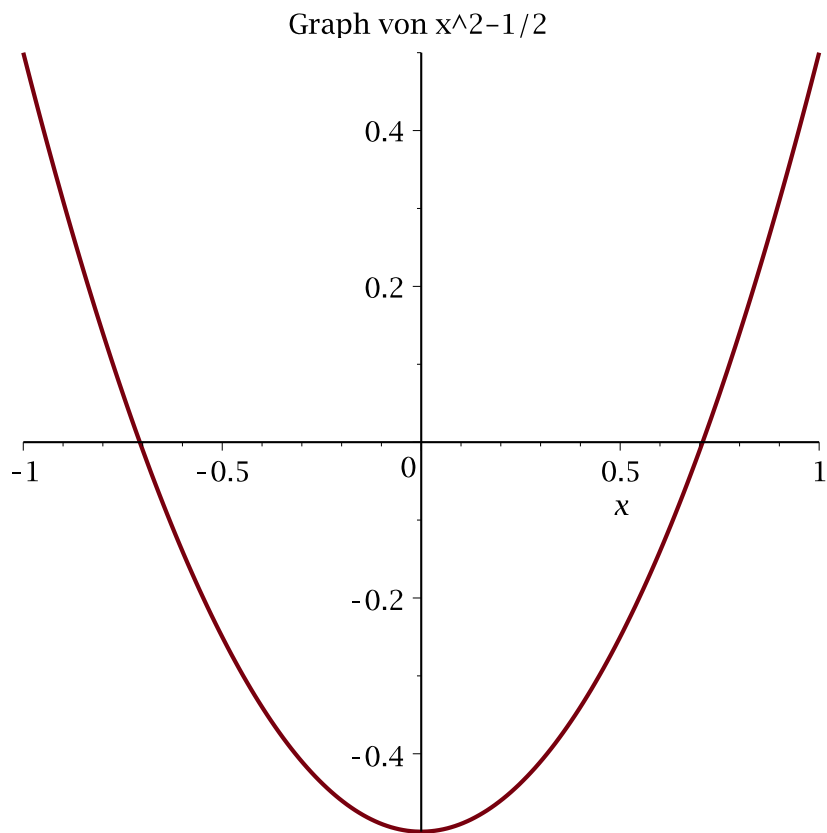
```

$$\tag{3.10}$$

```

> plot(f, x = -1 .. 1, title = Beschreibung, thickness = 2);

```



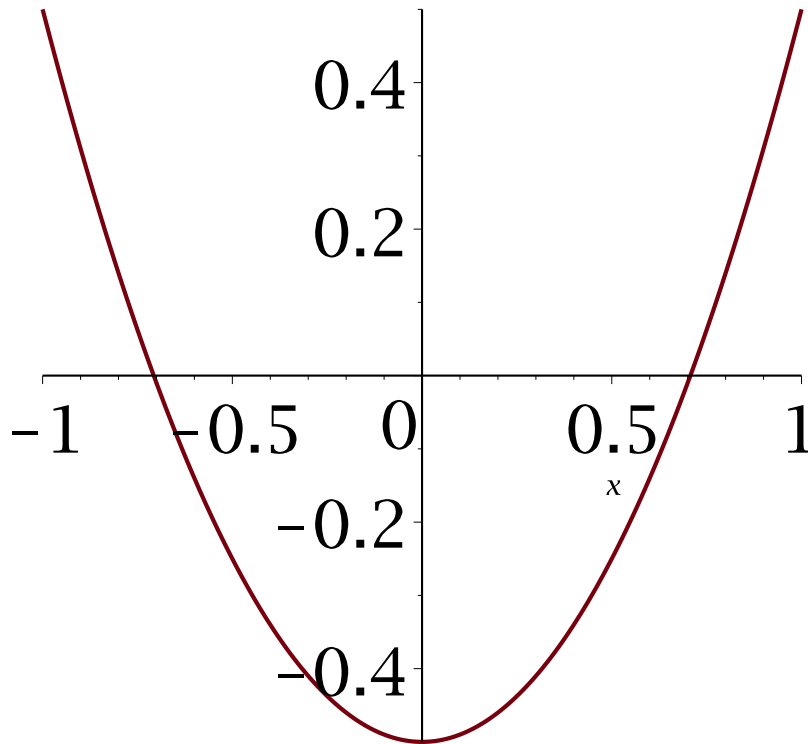
```
> Font := [TIMES, ROMAN, 24];
```

```
Font := [TIMES, ROMAN, 24]
```

(3.11)

```
> plot(f, x = -1 .. 1, title = Beschreibung, font = Font,  
thickness = 2);
```

## Graph von $x^2 - 1/2$



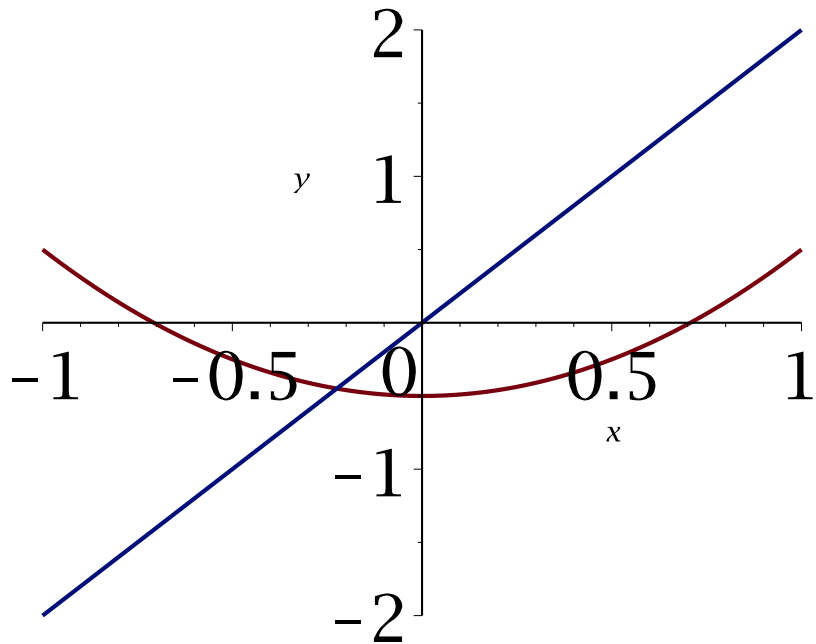
```
> Beschreibung2 := cat( Beschreibung, " und ihrer Ableitung");  
  Beschreibung2 := "Graph von  $x^2 - 1/2$  und ihrer Ableitung" (3.12)
```

```
> Optionen := title = Beschreibung2, font = Font, titlefont =  
  Font, labels = [x,y], thickness = 2;  
Optionen := title = "Graph von  $x^2 - 1/2$  und ihrer Ableitung", font = [TIMES, (3.13)  
  ROMAN, 24], titlefont = [TIMES, ROMAN, 24], labels = [x, y], thickness = 2
```

```
> pl1 := plot([f, diff(f, x)], x = -1 .. 1, Optionen);  
> pl1;
```

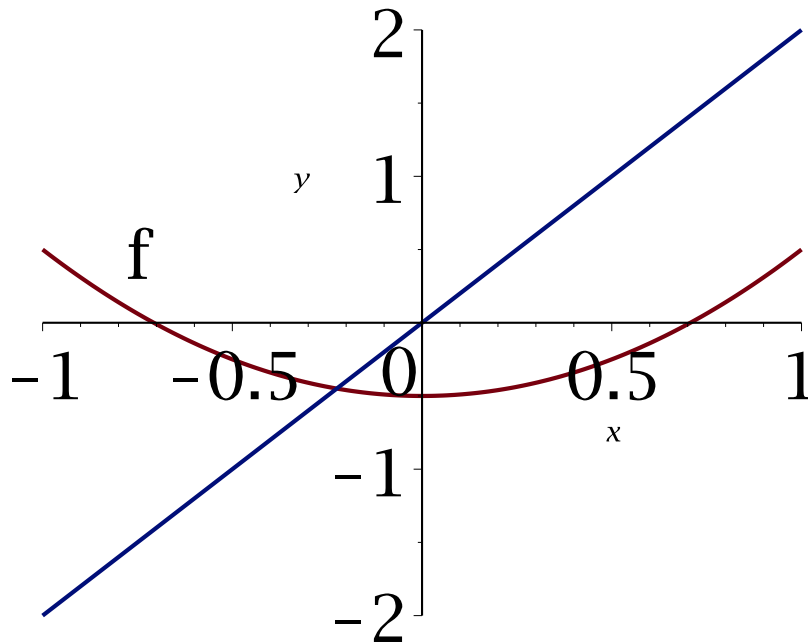


# Graph von $x^2 - 1/2$ und ihrer Ableitung



```
> with(plots):  
> t1 := textplot([-0.8, 0.2, "f"], align = {ABOVE, RIGHT}, font =  
  Font):  
> display({p1, t1});
```

# Graph von $x^2 - 1/2$ und ihrer Ableitung



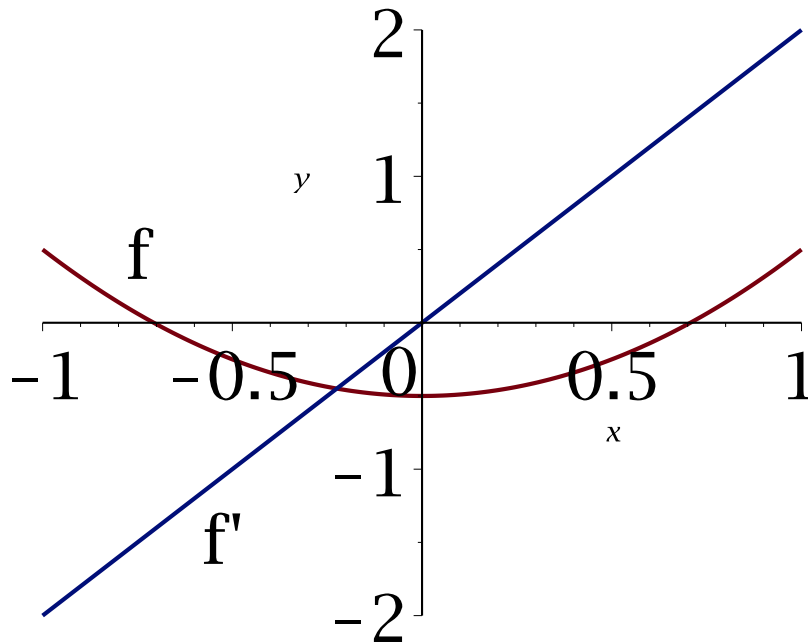
```
> position := -.6, subs(x = -.6, diff(f,x));  
           position := -0.6, -1.2
```

(3.14)

```
> t2 := textplot([position, "f'"], align = {BELOW, RIGHT}, font =  
Font):
```

```
> display({p11, t1, t2}, font = Font);
```

# Graph von $x^2 - 1/2$ und ihrer Ableitung



```
> t3 := textplot3d([1, 0, 1, "Text schwebt im Raum"], font =
  Font, color = green):
```

```
> x := r*cos(theta);
```

$$x := r \cos(\theta)$$

(3.15)

```
> y := r*sin(theta);
```

$$y := r \sin(\theta)$$

(3.16)

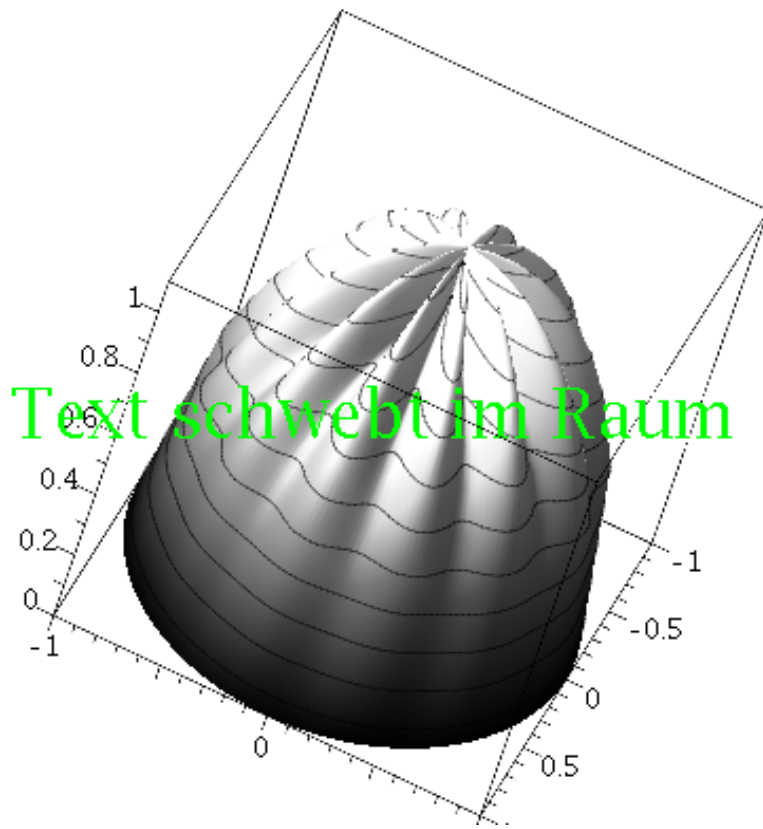
```
> f := sqrt(1 - r^2) * (1 - sin(12*theta)/10);
```

$$f := \sqrt{-r^2 + 1} \left( 1 - \frac{1}{10} \sin(12\theta) \right)$$

(3.17)

```
> pl2 := plot3d([x, y, f], r = 0 .. 1, theta = 0 .. 2*Pi,
  lightmodel = light4, shading = zgrayscale, grid = [10, 200],
  style = patchcontour):
```

```
> display(pl2, t3);
```



```
> dateiname := "test":  
> plotsetup(gif, plotoutput = dateiname);  
> display(pl2, t3);  
> plotsetup(window);  
> display(pl2, t3);
```