Computergestuetzte Mathematik zur Analysis

Lektion 10 (19. Dezember)

```
> restart: with(plots):
```

Partielle Ableitungen

```
> f := exp(x);
```

$$f := e^x \quad (1.1)$$

```
> df := Diff(f, x);
```

$$df := \frac{d}{dx} e^x \quad (1.2)$$

```
> value(df);
```

$$e^x \quad (1.3)$$

```
> g := exp(a*x + b*y + c*z);
```

$$g := e^{ax+by+cz} \quad (1.4)$$

```
> dg := Diff(g, x);
```

$$dg := \frac{\partial}{\partial x} e^{ax+by+cz} \quad (1.5)$$

```
> value(dg);
```

$$a\, e^{ax+by+cz} \quad (1.6)$$

```
> d123g := Diff(g, x, y, y, z$3):
> value(d123g);
```

$$a\, b^2\, c^3\, e^{ax+by+cz} \quad (1.7)$$

```
> h := (x, y, z) -> sin(a*x + b*y + c*z);
```

$$h := (x, y, z) \to \sin(a\,x + b\,y + c\,z) \quad (1.8)$$

```
> D[2](h);
```

$$(x, y, z) \to \cos(a\,x + b\,y + c\,z)\, b \quad (1.9)$$

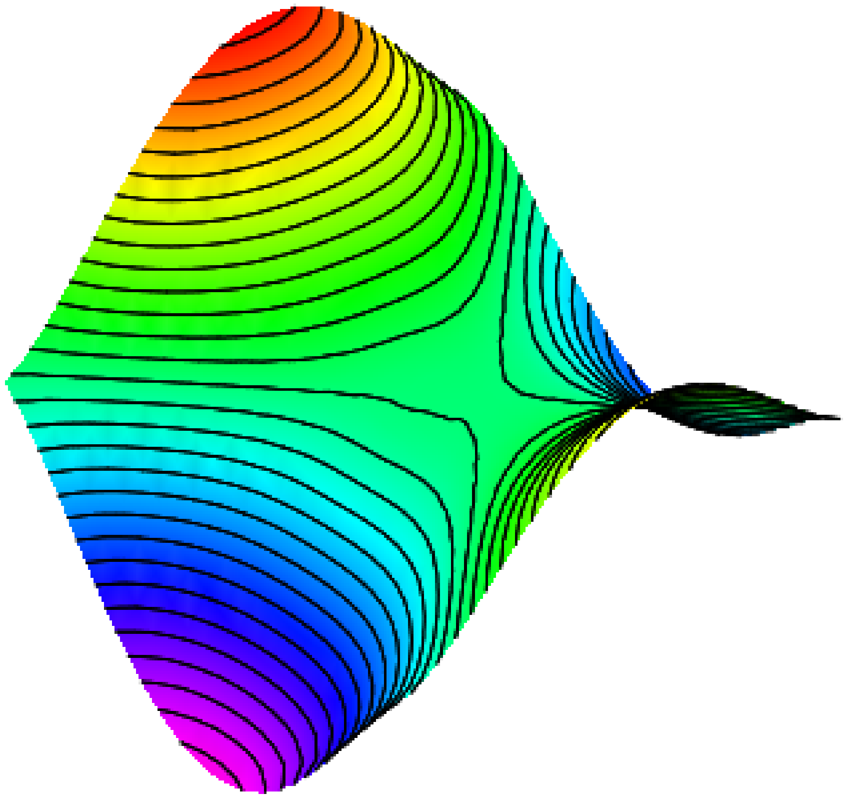
```
> D[1, 2, 2, 3$3](h);
```

$$(x, y, z) \to -\sin(a\,x + b\,y + c\,z)\, a\, b^2\, c^3 \quad (1.10)$$

```
> f := (x, y) -> sin(sqrt(x^2 + y^2)) * ((x-1/4)^2-(y-1/3)^2);
```

$$f := (x, y) \to \sin\left(\sqrt{x^2 + y^2}\right) \left(\left(x - \frac{1}{4}\right)^2 - \left(y - \frac{1}{3}\right)^2 \right) \quad (1.11)$$

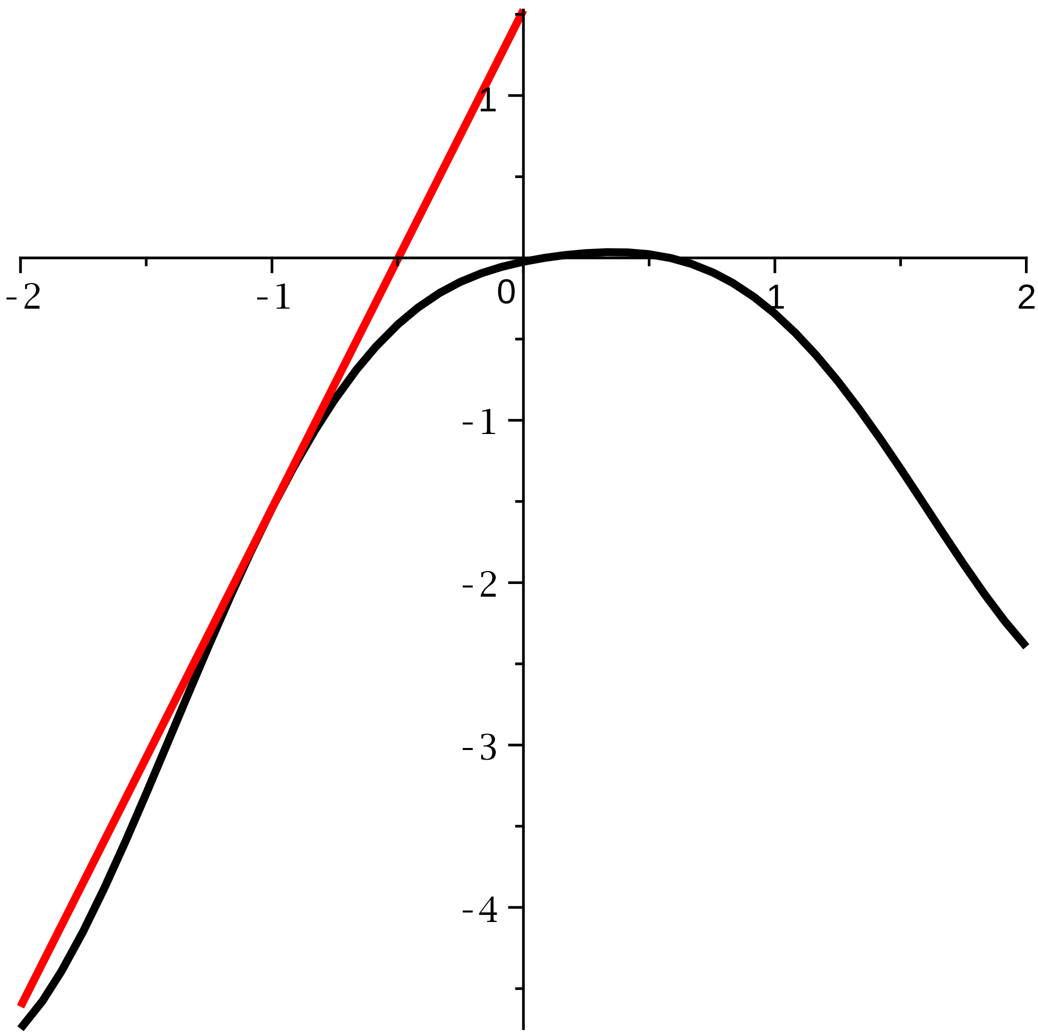
```
> p1 := plot3d(f-.05, -2 .. 2, -2 .. 2, style = surfacecontour,
  contours=30, shading = zhue):
> display(p1,orientation=[-40,50]);
```



```
> y_schnittkurve := [t, y, f(t, y), y = -2..2];
```

$$y\_schnittkurve := \left[t, y, \sin\left(\sqrt{t^2+y^2}\right)\left(\left(t-\frac{1}{4}\right)^2-\left(y-\frac{1}{3}\right)^2\right), y=-2..2\right] \quad \textbf{(1.12)}$$

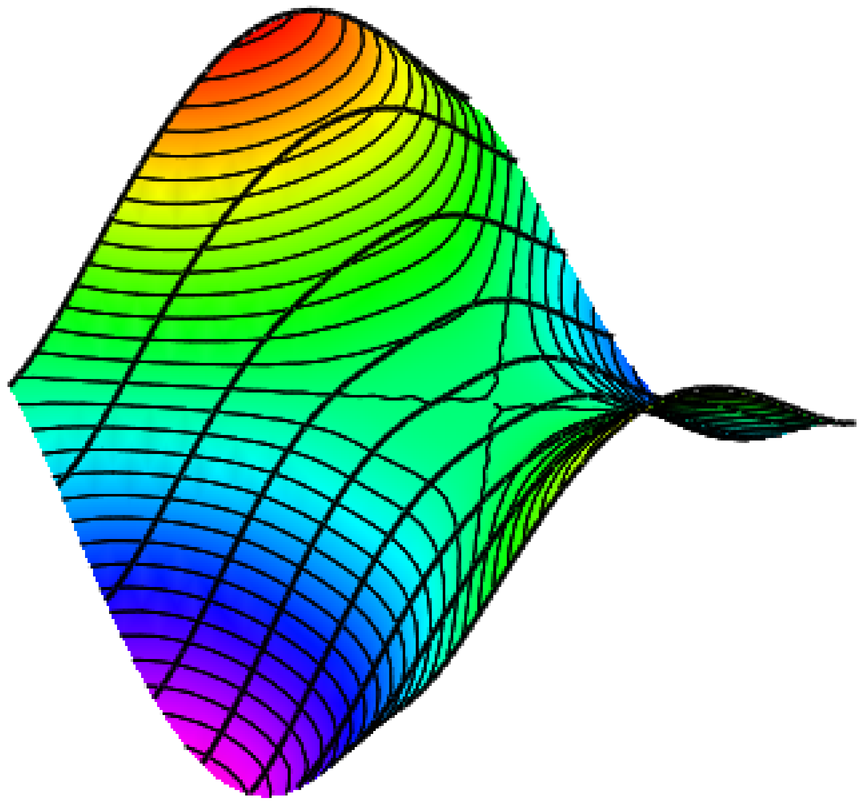
```
> tangente := f(1/2,-1) + D[2](f)(1/2,-1) + D[2](f)(1/2,-1)*y:
> plot([[y,f(1/2,y),y=-2..2],[y,tangente,y=-2..0]],color=
  [black,red], thickness = 3);
```




```
> y_schnitte :=spacecurve({seq(y_schnittkurve, t=-2..2,1/2)},
  color = black, thickness = 2);
```

$y\_schnitte := PLOT3D(...)$ **(1.13)**

```
> display([p1,y_schnitte],orientation=[-40,50]);
```



```
> p := <-3/2, -1, f(-3/2, -1)>;
```

$$p := \begin{bmatrix} -\frac{3}{2} \\ -1 \\ \frac{185}{144}\sin\left(\frac{1}{2}\sqrt{13}\right) \end{bmatrix} \tag{1.14}$$

```
> Dy := D[2](f)(-3/2, -1);
```

$$Dy := -\frac{185}{936}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} + \frac{8}{3}\sin\left(\frac{1}{2}\sqrt{13}\right) \tag{1.15}$$

```
> y_tan := p + t.<0,1,Dy>;
```

$$y\_tan := t.\begin{bmatrix} 0 \\ 1 \\ -\frac{185}{936}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} + \frac{8}{3}\sin\left(\frac{1}{2}\sqrt{13}\right) \end{bmatrix} \tag{1.16}$$

$$+\begin{bmatrix} -\frac{3}{2} \\ -1 \\ \frac{185}{144}\sin\left(\frac{1}{2}\sqrt{13}\right) \end{bmatrix}$$

```
> y_tan := simplify(y_tan);
Error, (in iroot) powering may produce overflow
> y_tan_pl := spacecurve(convert(y_tan, list), t = -1 .. 3/2,
   color = red, thickness = 3):
Error, (in plots/spacecurv) improper op or subscript selector
> display({p1,y_schnitte,y_tan_pl}, orientation=[-40,50]);
Error, (in plots:-display) expecting plot structures but
received: {y_tan_pl}
> grad := <D[1](f)(-3/2,-1),D[2](f)(-3/2,-1)>; ngrad := norm
   (grad,2): dgrad:= simplify(grad/ngrad):
```

$$grad := \begin{bmatrix} -\frac{185}{624}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} - \frac{7}{2}\sin\left(\frac{1}{2}\sqrt{13}\right) \\ -\frac{185}{936}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} + \frac{8}{3}\sin\left(\frac{1}{2}\sqrt{13}\right) \end{bmatrix} \quad (1.17)$$

```
> grad_tan := p+t.<dgrad[1],dgrad[2],ngrad>;
```

$$grad\_tan := t.\left[\left[-\left(3\left(185\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}+2184\sin\left(\frac{1}{2}\sqrt{13}\right)\right)\right)\Big/\left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2 + 3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right)+67848768\right)^{1/2}\right],\right.$$
$$\left[-\left(2\left(185\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}-2496\sin\left(\frac{1}{2}\sqrt{13}\right)\right)\right)\Big/\left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2 + 3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right)+67848768\right)^{1/2}\right],$$
$$\left.\left[\left(\left(\frac{185}{624}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}+\frac{7}{2}\sin\left(\frac{1}{2}\sqrt{13}\right)\right)^2+\left(-\frac{185}{936}\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}+\frac{8}{3}\sin\left(\frac{1}{2}\sqrt{13}\right)\right)^2\right)^{1/2}\right]\right] \quad (1.18)$$

$$+\begin{bmatrix} -\frac{3}{2} \\ -1 \\ \frac{185}{144}\sin\left(\frac{1}{2}\sqrt{13}\right) \end{bmatrix}$$

```
> grad_tan := simplify(grad_tan);
```

$$grad\_tan := \left[\left[-\frac{3}{2}\left(\left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2 + 3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right) + 67848768\right)^{1/2} + 370\,t\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} + 4368\,t\sin\left(\frac{1}{2}\sqrt{13}\right)\right) \Big/ \left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2 + 3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right) + 67848768\right)^{1/2}\right],\right.$$ **(1.19)**

$$\left[-\left(\left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2 + 3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right) + 67848768\right)^{1/2} + 370\,t\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13} - 4992\,t\sin\left(\frac{1}{2}\sqrt{13}\right)\right) \Big/ \left(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2\right.$$

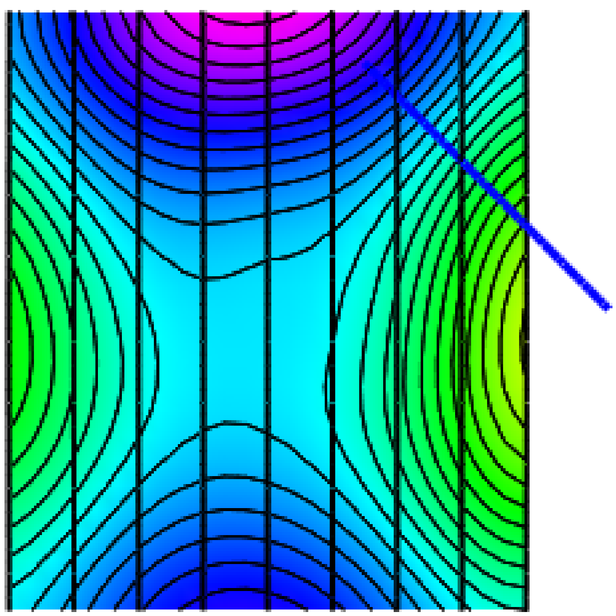
$$+3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right)+67848768\Big)^{1/2}\Bigg],$$

$$\left[\frac{185}{144}\sin\left(\frac{1}{2}\sqrt{13}\right)\right.$$

$$+\frac{1}{1872}\Bigg(-62064743\cos\left(\frac{1}{2}\sqrt{13}\right)^2$$

$$+3578640\cos\left(\frac{1}{2}\sqrt{13}\right)\sqrt{13}\sin\left(\frac{1}{2}\sqrt{13}\right)+67848768\Bigg)^{1/2}t\Bigg]\Bigg]$$

```
> grad_tan_pl :=spacecurve(convert(grad_tan, list), t = -1 ..
  3/2, color = blue, thickness = 3):
> display({p1,y_schnitte,grad_tan_pl}, orientation=[90,00]);
```



Ableitungen von Vektorfunktionen

```
> restart:
```

```
> v := <t, t^2, t^3>;
```

$$v := \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} \quad \mathbf{(2.1)}$$

```
> diff(v, t):
Error, non-algebraic expressions cannot be differentiated
> with(VectorCalculus):
> diff(v, t);
```

$$e_x + 2\,t e_y + 3\,t^2 e_z \quad \mathbf{(2.2)}$$

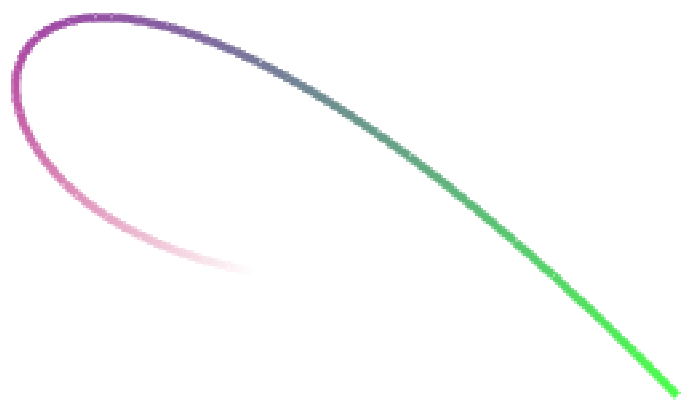
```
> BasisFormat(false);
```

true **(2.3)**

```
> dv := diff(v, t);
```

$$dv := \begin{bmatrix} 1 \\ 2\,t \\ 3\,t^2 \end{bmatrix} \quad \mathbf{(2.4)}$$

```
> with(plots):
> spacecurve(v, t = -3 .. 3,thickness=3);
```

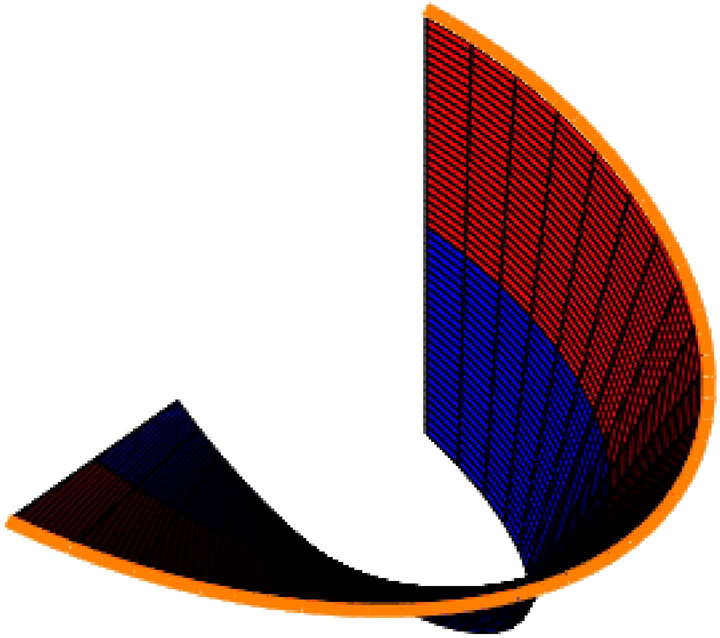


Moebiusband

```
> restart: with(plots):
> M := <cos(t)*(1 + s*cos(t/2)), sin(t)*(1+s*cos(t/2)), s*sin
  (t/2)>;
  p1:= plot3d(M, t = 0 .. Pi, s=-1/2..0,color=blue);
  p2:= plot3d(M, t = 0 .. Pi, s=0..1/2,color=red);
  p3:= spacecurve(subs(s=1/2+0.02,convert(M,list)),t=0..Pi,
  color=coral,thickness=5);
  display({p1,p2,p3});
```

$$M := \begin{bmatrix} \cos(t)\left(1 + s\cos\left(\frac{1}{2}\,t\right)\right) \\ \sin(t)\left(1 + s\cos\left(\frac{1}{2}\,t\right)\right) \\ s\sin\left(\frac{1}{2}\,t\right) \end{bmatrix}$$

$$p1 := PLOT3D(...)$$
$$p2 := PLOT3D(...)$$
$$p3 := PLOT3D(...)$$



```
> Seele := subs(s = 0, M);
```

$$Seele := \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix} \tag{3.1}$$

```
> with(VectorCalculus):
> BasisFormat(false);
```

$$true \tag{3.2}$$

```
> Mt := diff(Seele, t);
```

$$Mt := \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix} \tag{3.3}$$

```
> Ms := diff(M, s);
```

$$Ms := \begin{bmatrix} \cos(t)\cos\left(\frac{1}{2}t\right) \\ \sin(t)\cos\left(\frac{1}{2}t\right) \\ \sin\left(\frac{1}{2}t\right) \end{bmatrix} \quad \textbf{(3.4)}$$

```
> with(LinearAlgebra):
> Normale := CrossProduct(Ms, Mt);
```

$$Normale := \begin{bmatrix} -\sin\left(\frac{1}{2}t\right)\cos(t) \\ -\sin\left(\frac{1}{2}t\right)\sin(t) \\ \cos(t)^2\cos\left(\frac{1}{2}t\right) + \sin(t)^2\cos\left(\frac{1}{2}t\right) \end{bmatrix} \quad \textbf{(3.5)}$$

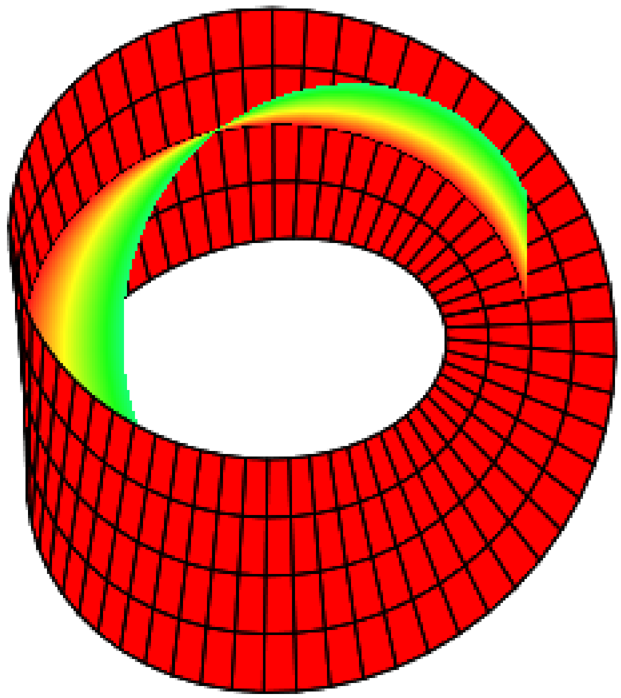
```
> pl1 := plot3d(M, t = 0 .. 2*Pi, s = -1/3 .. 1/3, grid = [60,
  5], color = red):
> EinheitsNormale := simplify(Normale/Norm(Normale, 2))
  assuming t::real:
> EinheitsNormale[1];
```

$$-\sin\left(\frac{1}{2}t\right)\left(2\cos\left(\frac{1}{2}t\right)^2 - 1\right) \quad \textbf{(3.6)}$$

```
> flaeche := convert(Seele + s*EinheitsNormale, list);
```

$$flaeche := \left[\cos(t) - s\sin\left(\frac{1}{2}t\right)\left(2\cos\left(\frac{1}{2}t\right)^2 - 1\right), \sin(t) - s\sin\left(\frac{1}{2}t\right)\sin(t), s\cos\left(\frac{1}{2}t\right)\right] \quad \textbf{(3.7)}$$

```
> pl2 := plot3d(flaeche, t = 0 .. 2*Pi, s = 0 .. .4, color = s,
  numpoints = 3000, style = patchnogrid):
> with(plots):
> display({pl1, pl2}, orientation = [-78, -159]);
```



Gradienten und Vektorfelder

```
> restart:
> with(VectorCalculus):
> BasisFormat(false);
```

$$true \quad (4.1)$$

```
> f := a*x^2 + b*y^2 + c*z^2;
```

$$f := a\,x^2 + b\,y^2 + c\,z^2 \quad (4.2)$$

```
> gr := Gradient(f, [x, y, z]);
```

$$gr := \begin{bmatrix} 2\,a\,x \\ 2\,b\,y \\ 2\,c\,z \end{bmatrix} \quad (4.3)$$

```
> gr . <b*y, -a*x,0>;
```

$$0 \quad (4.4)$$

```
> vf := VectorField(<b*y, -a*x, 0>, cartesian[x,y,z]);
```

$$vf := \begin{bmatrix} b\,y \\ -a\,x \\ 0 \end{bmatrix} \quad (4.5)$$

```
> gr . vf; # Skalarprodukt
```

$$0 \quad (4.6)$$

Zeichnungen von Vektorfeldern

```
> restart:
> with(VectorCalculus):
> BasisFormat(false);
```

$$true \quad (5.1)$$

```
> vf1 := VectorField(<-y, x>, cartesian[x,y]);
  vf2 := VectorField(<x, y>, cartesian[x,y]);
  vf3 := VectorField(<y, x>, cartesian[x,y]);
```

$$vf1 := \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$vf2 := \begin{bmatrix} x \\ y \end{bmatrix}$$

$$vf3 := \begin{bmatrix} y \\ x \end{bmatrix} \quad (5.2)$$

```
> with(plots):
> fieldplot(vf1, x = -1 .. 1, y = -1 .. 1, thickness = 2);
Error, (in plots/fieldplot) powering may produce overflow
> fieldplot(vf2, x=-1..1,y=-1..1,thickness=2);
Error, (in plots/fieldplot) powering may produce overflow
> fieldplot(vf3,x=-1..1,y=-1..1,thickness=2);
Error, (in plots/fieldplot) powering may produce overflow
> with(LinearAlgebra):
> vf2 := vf2/Norm(vf2, 2);
```

$$vf2 := \begin{bmatrix} \dfrac{x}{\sqrt{|x|^2 + |y|^2}} \\ \dfrac{y}{\sqrt{|x|^2 + |y|^2}} \end{bmatrix} \quad (5.3)$$

```
> fieldplot(vf2, x = -1 .. 1, y = -1 .. 1, thickness = 2, color
  = sqrt(x^2 + y^2));
Error, (in plots/fieldplot) powering may produce overflow
> k := -1/sqrt(x^2 + (y-1)^2 + 1) + 1/sqrt((x-1)^2 + (y+1)^2 +
```

```
1) + 1/sqrt((x+1)^2 + (y+1)^2 + 1);
```

$$k := -\frac{1}{\sqrt{x^2+y^2-2y+2}} + \frac{1}{\sqrt{x^2-2x+3+y^2+2y}} + \frac{1}{\sqrt{x^2+2x+3+y^2+2y}} \quad \textbf{(5.4)}$$

```
> gr := Gradient(k, [x,y]);
```

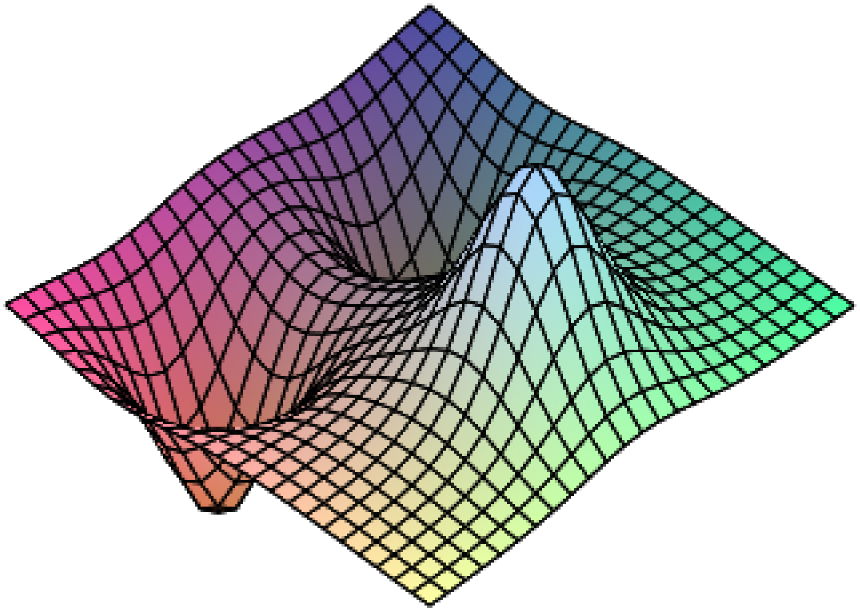
$$gr := \left[\left[\frac{x}{(x^2+y^2-2y+2)^{3/2}} - \frac{1}{2}\frac{2x-2}{(x^2-2x+3+y^2+2y)^{3/2}} - \frac{1}{2}\frac{2x+2}{(x^2+2x+3+y^2+2y)^{3/2}}\right], \left[\frac{1}{2}\frac{2y-2}{(x^2+y^2-2y+2)^{3/2}} - \frac{1}{2}\frac{2y+2}{(x^2-2x+3+y^2+2y)^{3/2}} - \frac{1}{2}\frac{2y+2}{(x^2+2x+3+y^2+2y)^{3/2}}\right]\right] \quad \textbf{(5.5)}$$

```
> fieldplot(gr, x = -2 .. 2, y = -2.3 .. 2.3, axes = frame,
  thickness = 2);
Error, (in plots/fieldplot) powering may produce overflow
```

```
> divgr :=Divergence(gr);
```

$$divgr := -\frac{3x^2}{(x^2+y^2-2y+2)^{5/2}} + \frac{2}{(x^2+y^2-2y+2)^{3/2}} + \frac{3}{4}\frac{(2x-2)^2}{(x^2-2x+3+y^2+2y)^{5/2}} - \frac{2}{(x^2-2x+3+y^2+2y)^{3/2}} + \frac{3}{4}\frac{(2x+2)^2}{(x^2+2x+3+y^2+2y)^{5/2}} - \frac{2}{(x^2+2x+3+y^2+2y)^{3/2}} - \frac{3}{4}\frac{(2y-2)^2}{(x^2+y^2-2y+2)^{5/2}} + \frac{3}{4}\frac{(2y+2)^2}{(x^2-2x+3+y^2+2y)^{5/2}} + \frac{3}{4}\frac{(2y+2)^2}{(x^2+2x+3+y^2+2y)^{5/2}} \quad \textbf{(5.6)}$$

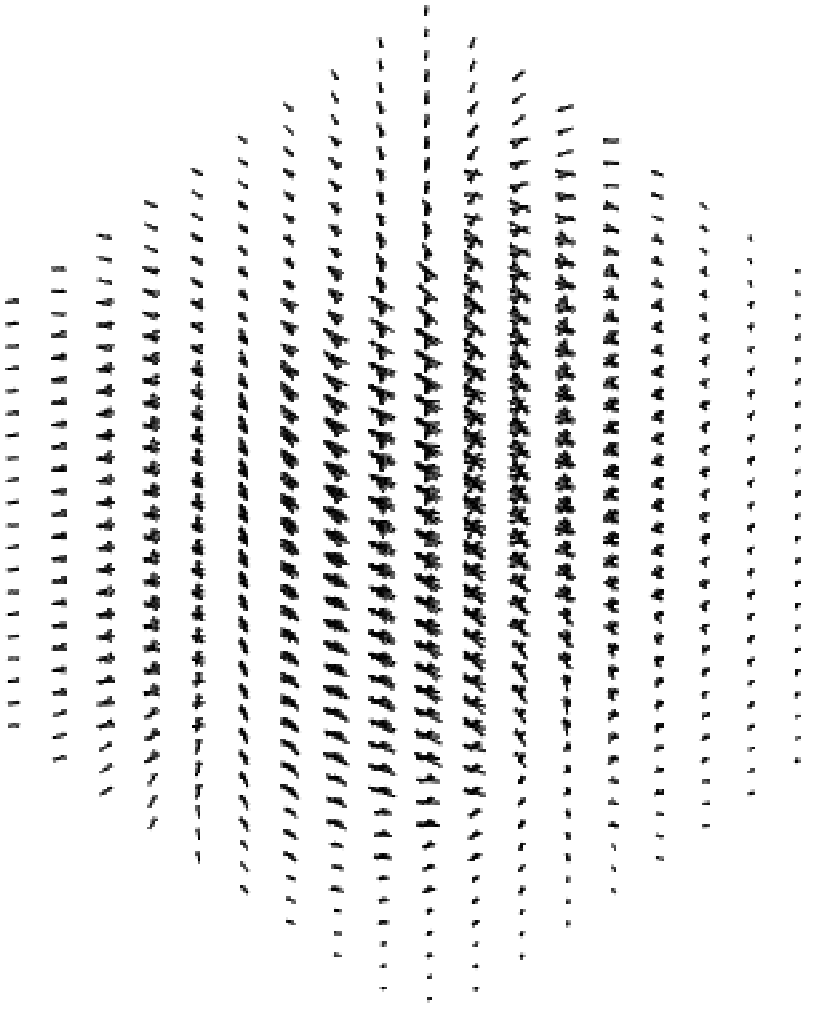
```
> plot3d(divgr,x=-2..2 ,y=-2..2);
```



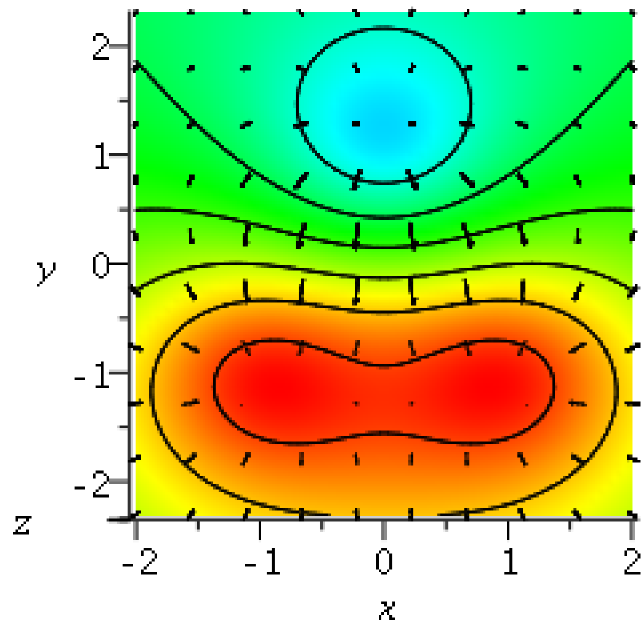
```
> gr3 := Gradient(k, [x,y,z]);
```

$$gr3 := \left[\left[\frac{x}{(x^2+y^2-2y+2)^{3/2}} - \frac{1}{2}\frac{2x-2}{(x^2-2x+3+y^2+2y)^{3/2}} - \frac{1}{2}\frac{2x+2}{(x^2+2x+3+y^2+2y)^{3/2}}\right], \left[\frac{1}{2}\frac{2y-2}{(x^2+y^2-2y+2)^{3/2}} - \frac{1}{2}\frac{2y+2}{(x^2-2x+3+y^2+2y)^{3/2}} - \frac{1}{2}\frac{2y+2}{(x^2+2x+3+y^2+2y)^{3/2}}\right], \left[0\right]\right] \quad (5.7)$$

```
> f3 := fieldplot3d(gr3, x = -2 .. 2, y = -2.3 .. 2.3, z = -1 .
  . 1, color = black, grid = [10, 10, 20]):
> f3;
```



```
> pl3 := plot3d(k, x = -2 .. 2, y = -2.3 .. 2.3, shading =
  zhue, style = patchcontour, numpoints = 3000):
> display({f3, pl3}, axes = frame, orientation = [-90,0]);
```



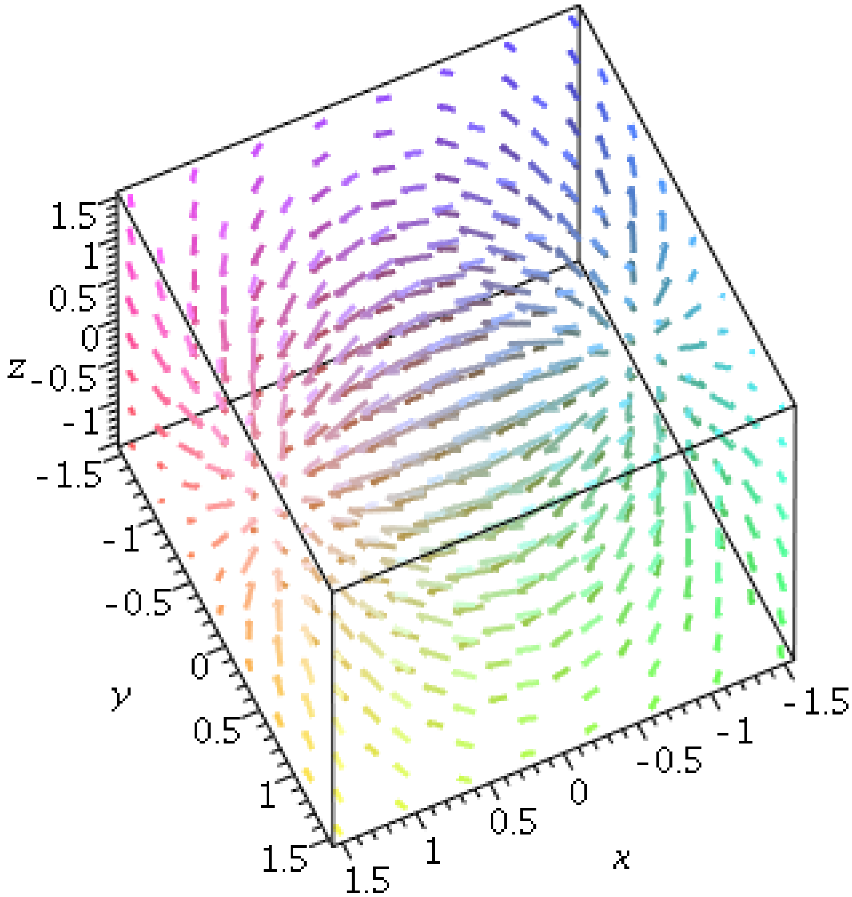

```
> k3 := 1/sqrt((x-1)^2 + y^2 + z^2 + 1) - 1/sqrt((x+1)^2 + y^2 + z^2 + 1);
```

$$k3 := \frac{1}{\sqrt{x^2 - 2x + 2 + y^2 + z^2}} - \frac{1}{\sqrt{x^2 + 2x + 2 + y^2 + z^2}} \quad (5.8)$$

```
> gr := Gradient(k3, [x,y,z]);
```

$$gr := \left[\begin{array}{c} -\frac{1}{2}\frac{2x-2}{(x^2-2x+2+y^2+z^2)^{3/2}} + \frac{1}{2}\frac{2x+2}{(x^2+2x+2+y^2+z^2)^{3/2}} \\ -\frac{y}{(x^2-2x+2+y^2+z^2)^{3/2}} + \frac{y}{(x^2+2x+2+y^2+z^2)^{3/2}} \\ -\frac{z}{(x^2-2x+2+y^2+z^2)^{3/2}} + \frac{z}{(x^2+2x+2+y^2+z^2)^{3/2}} \end{array}\right] \quad (5.9)$$

```
> fieldplot3d(gr, x = -1.5..1.5, y = -1.5..1.5, z = -1.5..1.5, orientation = [65, 30], axes = boxed, thickness = 2);
```



Divergenz und Rotation

```
> with(VectorCalculus):
> SetCoordinates(cartesian[x, y, z]):
> BasisFormat(false);
```

false **(5.1.1)**

```
> F := VectorField(<x*y,-y*z, z*z>);
```

$$F := \begin{bmatrix} x\,y \\ -y\,z \\ z^2 \end{bmatrix} \quad \textbf{(5.1.2)}$$

```
> Divergence(F);
```

$$y + z \quad \textbf{(5.1.3)}$$

```
> Divergence(Gradient(h(x, y, z)));
```

$$\frac{\partial^2}{\partial x^2}\,h(x, y, z) + \frac{\partial^2}{\partial y^2}\,h(x, y, z) + \frac{\partial^2}{\partial z^2}\,h(x, y, z) \quad \textbf{(5.1.4)}$$

```
> Laplacian(h(x, y, z));
```

$$\frac{\partial^2}{\partial x^2} h(x, y, z) + \frac{\partial^2}{\partial y^2} h(x, y, z) + \frac{\partial^2}{\partial z^2} h(x, y, z) \quad \textbf{(5.1.5)}$$

```
> E:= VectorField(<a(x,y,z),b(x,y,z),c(x,y,z)>);
```

$$E := \begin{bmatrix} a(x, y, z) \\ b(x, y, z) \\ c(x, y, z) \end{bmatrix} \quad \textbf{(5.1.6)}$$

```
> Curl(E); # Rotation  Gradient X E
```

$$\begin{bmatrix} \frac{\partial}{\partial y} c(x, y, z) - \left(\frac{\partial}{\partial z} b(x, y, z)\right) \\ \frac{\partial}{\partial z} a(x, y, z) - \left(\frac{\partial}{\partial x} c(x, y, z)\right) \\ \frac{\partial}{\partial x} b(x, y, z) - \left(\frac{\partial}{\partial y} a(x, y, z)\right) \end{bmatrix} \quad \textbf{(5.1.7)}$$

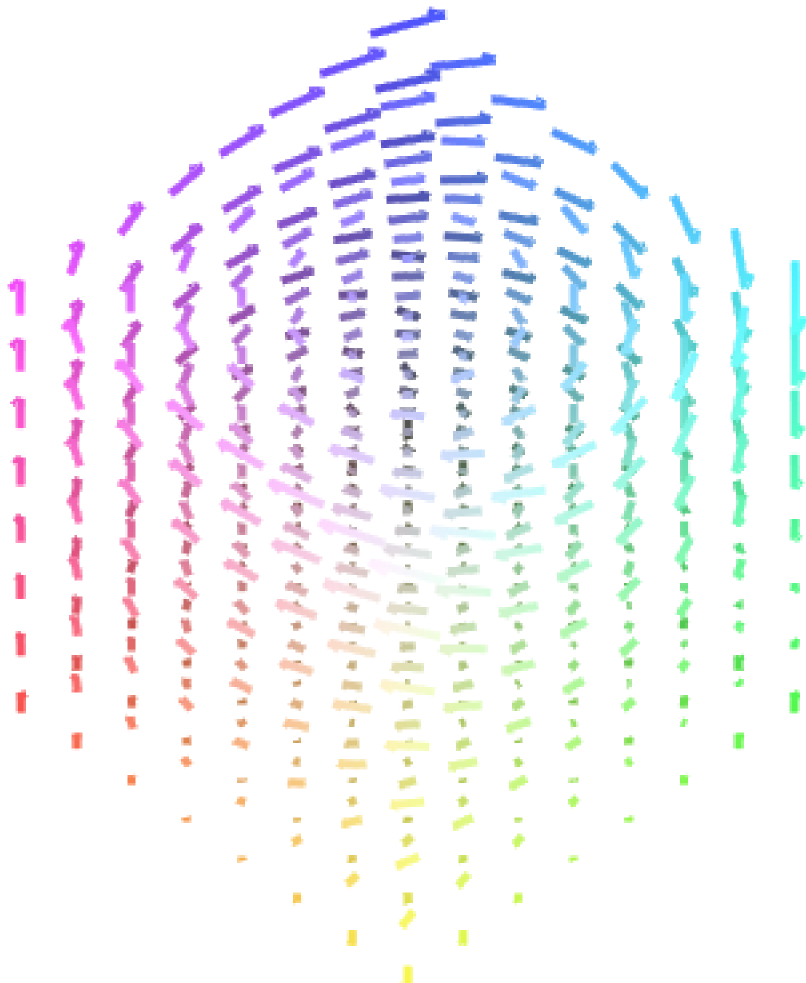
```
> Curl(gr);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(5.10)}$$

```
> vf := VectorField(<y+z*y,-x-z*x,x*y*z>,cartesian[x,y,z]);
```

$$vf := \begin{bmatrix} y + y\,z \\ -x - z\,x \\ x\,y\,z \end{bmatrix} \quad \textbf{(5.11)}$$

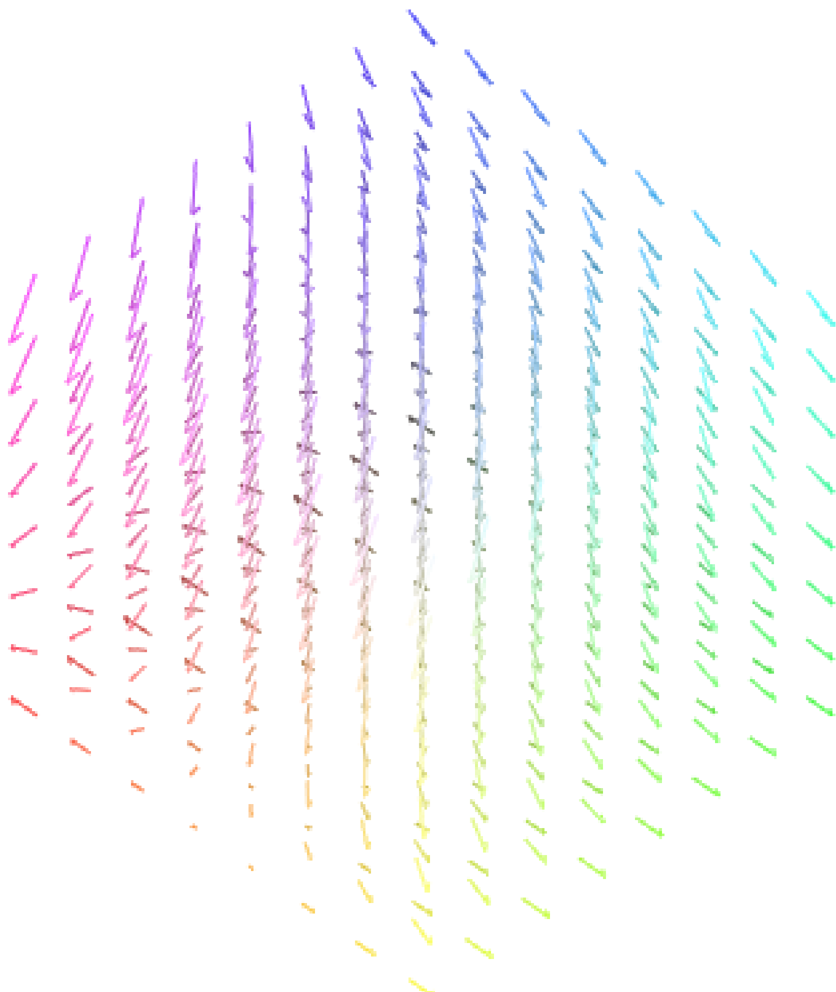
```
> fieldplot3d(vf,x=-1..1,y=-1..1,z=-1..1,thickness=3);
```



```
> Curl(vf);
```

$$\begin{bmatrix} z\,x+x \\ y-y\,z \\ -2-2\,z \end{bmatrix} \tag{5.12}$$

```
> fieldplot3d(Curl(vf),x=-1..1,y=-1..1,z=-1..1);
```



Jacobimatrix

```
> restart:
> with(VectorCalculus):
> BasisFormat(false):
> F := <F1(x,y,z), F2(x,y,z)>;
```

$$F := \begin{bmatrix} F1(x, y, z) \\ F2(x, y, z) \end{bmatrix} \tag{6.1}$$

```
> Jacobian(F, [x,y,z]);
```

$$\begin{bmatrix} \frac{\partial}{\partial x} F1(x, y, z) & \frac{\partial}{\partial y} F1(x, y, z) & \frac{\partial}{\partial z} F1(x, y, z) \\ \frac{\partial}{\partial x} F2(x, y, z) & \frac{\partial}{\partial y} F2(x, y, z) & \frac{\partial}{\partial z} F2(x, y, z) \end{bmatrix} \tag{6.2}$$

```
> F3 := <F[1], F[2], 0>;
```

$$F3 := \begin{bmatrix} F1(x, y, z) \\ F2(x, y, z) \\ 0 \end{bmatrix} \quad (6.3)$$

```
> Jacobian(F3, [x,y,z]);
```

$$\begin{bmatrix} \frac{\partial}{\partial x} F1(x, y, z) & \frac{\partial}{\partial y} F1(x, y, z) & \frac{\partial}{\partial z} F1(x, y, z) \\ \frac{\partial}{\partial x} F2(x, y, z) & \frac{\partial}{\partial y} F2(x, y, z) & \frac{\partial}{\partial z} F2(x, y, z) \\ 0 & 0 & 0 \end{bmatrix} \quad (6.4)$$

```
> with(LinearAlgebra):
> jac := SubMatrix(%%, 1..2, 1..3);
```

$$jac := \begin{bmatrix} \frac{\partial}{\partial x} F1(x, y, z) & \frac{\partial}{\partial y} F1(x, y, z) & \frac{\partial}{\partial z} F1(x, y, z) \\ \frac{\partial}{\partial x} F2(x, y, z) & \frac{\partial}{\partial y} F2(x, y, z) & \frac{\partial}{\partial z} F2(x, y, z) \end{bmatrix} \quad (6.5)$$

```
> F := <x^2 + 2*x + 2 + y^2 - 2*y, x^2 + 2*x - y^2 + 2*y, x*y -
  x + y -1>;
```

$$F := \begin{bmatrix} x^2 + 2x + 2 + y^2 - 2y \\ x^2 + 2x - y^2 + 2y \\ xy - x + y - 1 \end{bmatrix} \quad (6.6)$$

```
> Jacobian(F, [x,y,z]);
```

$$\begin{bmatrix} 2x + 2 & 2y - 2 & 0 \\ 2x + 2 & -2y + 2 & 0 \\ y - 1 & x + 1 & 0 \end{bmatrix} \quad (6.7)$$

```
> J := SubMatrix(%, 1..3, 1..2);
```

$$J := \begin{bmatrix} 2x + 2 & 2y - 2 \\ 2x + 2 & -2y + 2 \\ y - 1 & x + 1 \end{bmatrix} \quad (6.8)$$

```
> Rank(J); # Vorsicht falsch
```

$$2 \quad (6.9)$$

```
> ReducedRowEchelonForm(J);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (6.10)$$

```
> J;
```

$$\begin{bmatrix} 2x+2 & 2y-2 \\ 2x+2 & -2y+2 \\ y-1 & x+1 \end{bmatrix} \quad (6.11)$$

```
> J1 := RowOperation(J, [2,1], -1);
```

$$J1 := \begin{bmatrix} 2x+2 & 2y-2 \\ 0 & -4y+4 \\ y-1 & x+1 \end{bmatrix} \quad (6.12)$$

```
> J2 := RowOperation(J1, [3,2]);
```

$$J2 := \begin{bmatrix} 2x+2 & 2y-2 \\ y-1 & x+1 \\ 0 & -4y+4 \end{bmatrix} \quad (6.13)$$

```
> J3 := RowOperation(J2, 1, y-1);   # ausser fuer y = 1
```

$$J3 := \begin{bmatrix} (y-1)(2x+2) & (2y-2)(y-1) \\ y-1 & x+1 \\ 0 & -4y+4 \end{bmatrix} \quad (6.14)$$

```
> J4 := RowOperation(J3, 2, 2*x+2);   # ausser fuer x = -1;
```

$$J4 := \begin{bmatrix} (y-1)(2x+2) & (2y-2)(y-1) \\ (y-1)(2x+2) & (x+1)(2x+2) \\ 0 & -4y+4 \end{bmatrix} \quad (6.15)$$

```
> RowOperation(J4, [2,1], -1);
```

$$\begin{bmatrix} (y-1)(2x+2) & (2y-2)(y-1) \\ 0 & (x+1)(2x+2)-(2y-2)(y-1) \\ 0 & -4y+4 \end{bmatrix} \quad (6.16)$$

```
> J5 := map(expand, %);
```

$$J5 := \begin{bmatrix} 2xy+2y-2x-2 & 2y^2-4y+2 \\ 0 & 2x^2+4x-2y^2+4y \\ 0 & -4y+4 \end{bmatrix} \quad (6.17)$$

```
> map(factor, J5);
```

$$\begin{bmatrix} 2(x+1)(y-1) & 2(y-1)^2 \\ 0 & 2(y+x)(x+2-y) \\ 0 & -4y+4 \end{bmatrix} \quad (6.18)$$

Also ist fuer x <>-1 und y<>1 der Rang tatsaechlich 2. Wir testen den Extremfall

```
> subs(x = -1, y = 1, J);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (6.19)$$

Hessematrix

f : R^n --> R

```
> with(VectorCalculus):
> f := (x, y, z) -> exp(x^2+y^2+z);
```

$$f := (x, y, z) \rightarrow e^{x^2+y^2+z} \quad (7.1)$$

```
> Hessian(f(x, y, z), [x, y, z]);
```

$$\begin{bmatrix} 2\,e^{x^2+y^2+z} + 4\,x^2\,e^{x^2+y^2+z} & 4\,x\,y\,e^{x^2+y^2+z} & 2\,x\,e^{x^2+y^2+z} \\ 4\,x\,y\,e^{x^2+y^2+z} & 2\,e^{x^2+y^2+z} + 4\,y^2\,e^{x^2+y^2+z} & 2\,y\,e^{x^2+y^2+z} \\ 2\,x\,e^{x^2+y^2+z} & 2\,y\,e^{x^2+y^2+z} & e^{x^2+y^2+z} \end{bmatrix} \quad (7.2)$$

```
> g := exp(x^2+y^2+z);
```

$$g := e^{x^2+y^2+z} \quad (7.3)$$

```
> Hessian(g, [x, y, z]);
```

$$\begin{bmatrix} 2\,e^{x^2+y^2+z} + 4\,x^2\,e^{x^2+y^2+z} & 4\,x\,y\,e^{x^2+y^2+z} & 2\,x\,e^{x^2+y^2+z} \\ 4\,x\,y\,e^{x^2+y^2+z} & 2\,e^{x^2+y^2+z} + 4\,y^2\,e^{x^2+y^2+z} & 2\,y\,e^{x^2+y^2+z} \\ 2\,x\,e^{x^2+y^2+z} & 2\,y\,e^{x^2+y^2+z} & e^{x^2+y^2+z} \end{bmatrix} \quad (7.4)$$

```
> with(LinearAlgebra):
> IsDefinite(subs([x = 1, y = 2, z = 1], (7.4)));
```

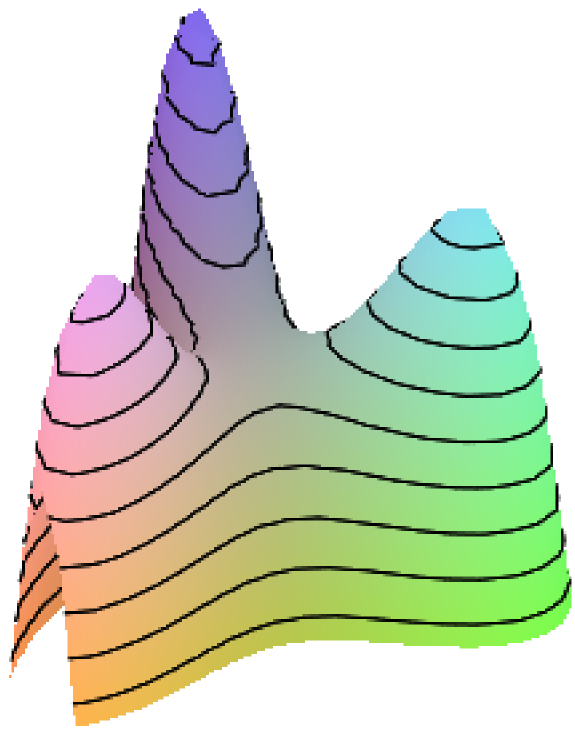
true (7.5)

Lokale Extrema

```
> restart;
> with(VectorCalculus): with(LinearAlgebra):
> f:= -1/2*x^4 - x^2*y^2-1/2*y^4+x^3-3*x*y^2;
```

$$f := -\frac{1}{2}\,x^4 - x^2\,y^2 - \frac{1}{2}\,y^4 + x^3 - 3\,x\,y^2 \quad (8.1)$$

```
> plot3d(f,x=-2..2,y=-2..2,view=-1..1,style=patchcontour);
```



```
> g:=Gradient(f,[x,y]);
```

$$g := \left(-2\,x^3 - 2\,x\,y^2 + 3\,x^2 - 3\,y^2\right)\bar{e}_x + \left(-2\,x^2\,y - 2\,y^3 - 6\,x\,y\right)\bar{e}_y \quad \textbf{(8.2)}$$

```
> H:=Hessian(f,[x,y]);
```

$$H := \begin{bmatrix} -6\,x^2 - 2\,y^2 + 6\,x & -4\,x\,y - 6\,y \\ -4\,x\,y - 6\,y & -2\,x^2 - 6\,y^2 - 6\,x \end{bmatrix} \quad \textbf{(8.3)}$$

```
> solve(convert(g,set),{x,y});
```

$$\{x = 0, y = 0\}, \{x = 0, y = 0\}, \left\{x = \frac{3}{2}, y = 0\right\}, \{x = 0, y = 0\}, \left\{x = -\frac{3}{4}, y = \frac{3}{4}\,RootOf\left(\_Z^2 - 3, label = \_L3\right)\right\} \quad \textbf{(8.4)}$$

```
> L:=solve(convert(g,set),{x,y},Explicit,DropMultiplicity);
```

$$L := \{x = 0, y = 0\}, \left\{x = \frac{3}{2}, y = 0\right\}, \left\{x = -\frac{3}{4}, y = \frac{3}{4}\sqrt{3}\right\}, \left\{x = -\frac{3}{4}, y = -\frac{3}{4}\sqrt{3}\right\} \quad \textbf{(8.5)}$$

```
> HH := seq(subs(L[k],H),k=1..4);
> IsDefinite(HH[2]);
```

$$HH := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{9}{2} & 0 \\ 0 & -\frac{27}{2} \end{bmatrix}, \begin{bmatrix} -\frac{45}{4} & -\frac{9}{4}\sqrt{3} \\ -\frac{9}{4}\sqrt{3} & -\frac{27}{4} \end{bmatrix}, \begin{bmatrix} -\frac{45}{4} & \frac{9}{4}\sqrt{3} \\ \frac{9}{4}\sqrt{3} & -\frac{27}{4} \end{bmatrix}$$

false **(8.6)**

```
> IsDefinite(HH[2],query=negative_definite);
```

true **(8.7)**

```
> IsDefinite(HH[3],query=negative_definite);
```

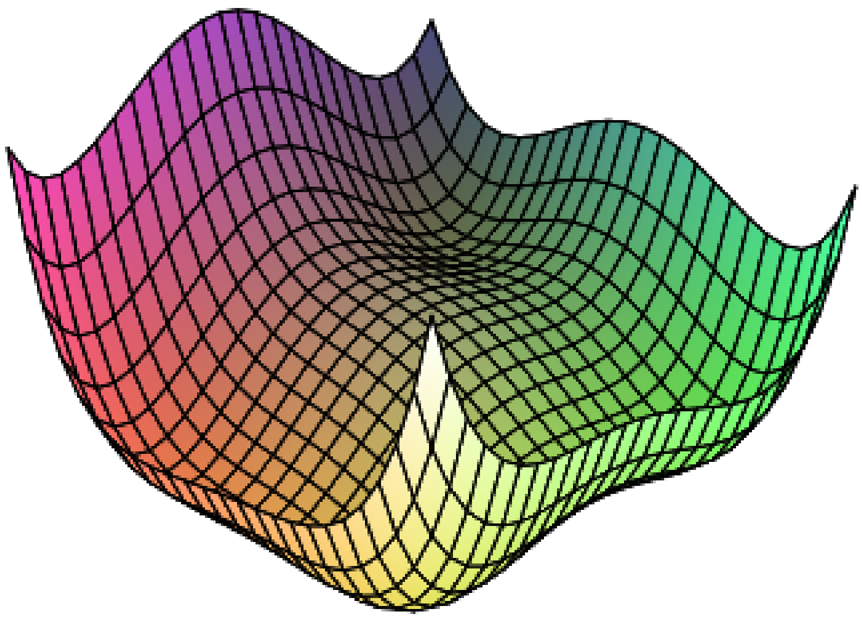
true **(8.8)**

Noch ein Beispiel

```
> f := (x^2+y-11)^2 + (x+y^2-7)^2;
```

$$f := \left(x^2 + y - 11\right)^2 + \left(x + y^2 - 7\right)^2 \quad \textbf{(8.9)}$$

```
> plot3d(f,x=-5..5,y=-5..5); # Himmelblaufunktion
```



```
> BasisFormat(false):
> g:=Gradient(f,[x,y]);
```

$$g := \begin{bmatrix} 4\left(x^2+y-11\right)x+2x+2y^2-14 \\ 2x^2+2y-22+4\left(x+y^2-7\right)y \end{bmatrix} \quad \textbf{(8.10)}$$

```
> H:=map(factor,Hessian(f,[x,y]));
```

$$H := \begin{bmatrix} 12x^2+4y-42 & 4x+4y \\ 4x+4y & -26+12y^2+4x \end{bmatrix} \quad \textbf{(8.11)}$$

```
> _EnvAllSolutions := true;
```

$$\_EnvAllSolutions := true \quad \textbf{(8.12)}$$

```
> L:=solve([g[1]=0,g[2]=0],{x,y});
```

$$L := \{x=3, y=2\}, \left\{x = -RootOf\left(\_Z^3+2\_Z^2-10\_Z-19\right)^2+7, y = RootOf\left(\_Z^3+2\_Z^2-10\_Z-19\right)\right\}, \left\{x = 13\, RootOf\left(\_Z^5-26\_Z^3\right.\right. \quad \textbf{(8.13)}$$

$$-42\,\_Z^2+1,\ label=\_L5)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1,\ label=\_L5)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1,\ label=\_L5),\ y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1,\ label=\_L5)\Big\}$$

```
> AVL := seq(allvalues(L[k]),k=1..3);
```

$$AVL:=\{x=3,\ y=2\},\ \left\{x=-\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}+\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}\right)^2+7,\ y=\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}+\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}\right\},\ \left\{x=-\left(-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right)^2+7,\ y=-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right\},\ \left\{x=-\left(-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}+\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right)^2+7,\ y=-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}\qquad\textbf{(8.14)}$$

$$+\frac{1}{2}\,I\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,I\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,I\sqrt{6303}\right)^{1/3}}\right)\Bigg\}, \Big\{x$$

$$= 13\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=1)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=1)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=1), y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=1)\Big\}, \Big\{x$$

$$= 13\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=2)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=2)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=2), y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=2)\Big\}, \Big\{x$$

$$= 13\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=3)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=3)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=3), y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=3)\Big\}, \Big\{x$$

$$= 13\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=4)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=4)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=4), y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=4)\Big\}, \Big\{x$$

$$= 13\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=5)^2-\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=5)^4+21\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=5), y=\frac{1}{2}\,RootOf(\_Z^5-26\,\_Z^3-42\,\_Z^2+1, index=5)\Big\}$$

```
> seq(simplify(evalc(AVL[k])) , k=1..7);
Error, (in iroot) powering may produce overflow
> seq(simplify(evalc(subs(AVL[k],g))) , k=1..7);
```

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix} \quad \textbf{(8.15)}$$

```
> L1:=L[1];
```

(8.16)

$$L1 := \{x = 3, y = 2\} \quad \mathbf{(8.16)}$$

```
> L2_ := allvalues(L[2]);
```

$$L2\_ := \left\{x = -\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}+\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}\right)^{2} + 7,\ y = \left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}+\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}\right\}, \left\{x = -\left(-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right)^{2} + 7,\ y = -\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right\}, \left\{x = -\left(-\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}+\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right)^{2} + 7,\ y = -\frac{1}{2}\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{17}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}-\frac{2}{3}+\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}-\frac{34}{9\left(\frac{317}{54}+\frac{1}{18}\,\mathrm{I}\sqrt{6303}\right)^{1/3}}\right)\right\} \quad \mathbf{(8.17)}$$

```
> L2[1] := {simplify(evalc(L2_[1][1])), simplify(evalc(L2_[1]
  [2]))};
> L2[2] := {simplify(evalc(L2_[2][1])), simplify(evalc(L2_[2]
  [2]))};
```

Error, (in iroot) powering may produce overflow

Error, (in iroot) powering may produce overflow

```
> L2[3] := {simplify(evalc(L2_[2][1])), simplify(evalc(L2_[2][2]))};
```

$$L2_3 := \left\{x = -\frac{43}{9} - \frac{4}{9}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right) + \frac{68}{9}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)^2 + \frac{4}{9}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right) + \frac{68}{9}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\sqrt{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right), y = -\frac{1}{3}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right) - \frac{2}{3} + \frac{1}{3}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\right\} \qquad \textbf{(8.18)}$$

```
> L3 := evalf(allvalues(L[3]));
```

$$L3 := \{x = 3.385154184, y = 0.07385187985\}, \{x = 0.0866778, y = 2.884254701\}, \{x = -3.073025752, y = -0.08135304430\}, \{x = -0.27084459, y = -0.9230385565\}, \{x = -0.12796136, y = -1.953714980\} \qquad \textbf{(8.19)}$$

```
> subs(L[1],H);
```

$$\begin{bmatrix} 74 & 20 \\ 20 & 34 \end{bmatrix} \qquad \textbf{(8.20)}$$

```
> IsDefinite((8.20));
```

true **(8.21)**

```
> subs(L2[1],H);
Error, invalid input: subs received L2[1], which is not valid for its 1st argument
> IsDefinite(??);
Error, Invalid label reference
> subs(L2[2],H);
Error, invalid input: subs received L2[2], which is not valid for its 1st argument
> IsDefinite(??);
Error, Invalid label reference
> subs(L2[3],H);
```

$$\left[\left[12\left(-\frac{43}{9} - \frac{4}{9}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right) + \frac{68}{9}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)^2\right.\right.\right. \qquad \textbf{(8.22)}$$

$$+\frac{4}{9}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{68}{9}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\sqrt{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\right.\right.$$

$$\left.\left.\sqrt{2101}\right)\right)\Bigg)^2-\frac{4}{3}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)-\frac{134}{3}$$

$$+\frac{4}{3}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right),\ -\frac{196}{9}$$

$$-\frac{28}{9}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)^2$$

$$+\frac{28}{9}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\sqrt{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\right.\right.$$

$$\left.\left.\sqrt{2101}\right)\right)\Bigg],$$

$$\Bigg[-\frac{196}{9}-\frac{28}{9}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)^2$$

$$+\frac{28}{9}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\sqrt{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\right.\right.$$

$$\left.\left.\sqrt{2101}\right)\right),\ -\frac{406}{9}+12\left(-\frac{1}{3}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\right.$$

$$\left.-\frac{2}{3}+\frac{1}{3}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\right)^2$$

$$-\frac{16}{9}\sqrt{34}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)^2$$

$$+\frac{16}{9}\sqrt{3}\sqrt{34}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)$$

$$+\frac{272}{9}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\sqrt{2101}\right)\right)\sqrt{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{317}\sqrt{3}\right.\right.$$

$\sqrt{2101}\Big)\Big)\Big]\Big]$

```
> IsDefinite((8.22));
```

$$true \qquad (8.23)$$

```
> seq(IsDefinite(subs(L3[k],H),query=negative_definite),k=1..5)
;
```

$$false, false, false, true, false \qquad (8.24)$$

```
> seq(IsDefinite(subs(L3[k],H),query=positive_definite),k=1..5)
;
```

$$false, false, false, false, false \qquad (8.25)$$

```
> seq(IsDefinite(subs(L3[k],H),query=positive_semidefinite),k=
1..5);
```

$$false, false, false, false, false \qquad (8.26)$$

```
> seq(IsDefinite(subs(L3[k],H),query=negative_semidefinite),k=
1..5);
```

$$false, false, false, true, false \qquad (8.27)$$

```
> seq(IsDefinite(subs(L3[k],H),query=indefinite),k=1..5);
```

$$true, true, true, false, true \qquad (8.28)$$